Honors Analysis
"the *last* sked" 2003

Show $y = e^x$

$\ln y = x \ln e$

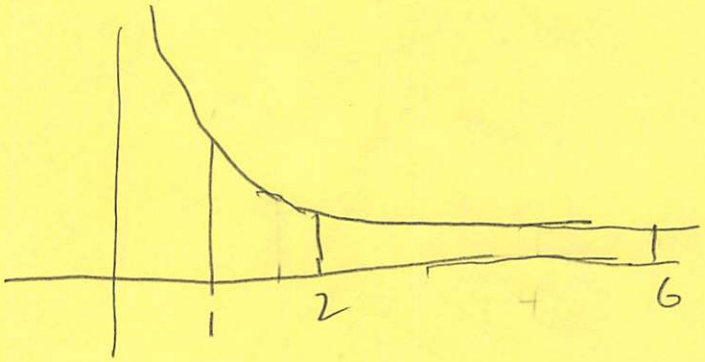
$\frac{1}{y}\frac{dy}{dx} = 1$

$\frac{dy}{dx} = y = e^x$

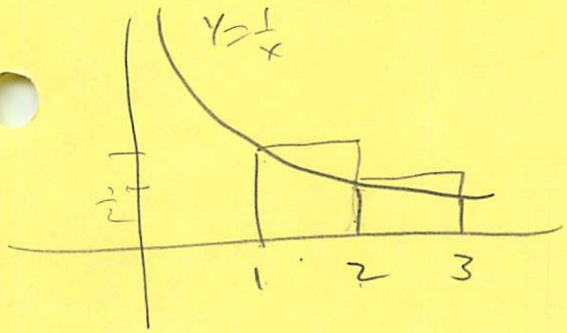
EXAM CHAPTER 4
Applications
5/16

# 1 R/C Pg 197 - 201 • • • • • • • • • • • • • • # 2 Pg 201–202 1 - 29 odd 5 / 19	# 3 Pg 201 - 202 2 - 28 even **Exponent Practice** 5/ 20	**Important Lecture!** # 4 Pg 207 1 - 10 Pg 213 1 - 21 (use Mumaugh Tables) and get decimal answers too. also 23, 24 **Area under curve Define fctn A(x) Intro to integral Define fctn and der of L(x)**	**Important Lecture** # 5 Pg 218 14 - 20 **last practice with L(x)** 5/ 22	**Important Lecture** # 6 Pg 222 1 -5, 6-9 find $g'(x)$, 10 - 12 Pg 224 1 - 7 **Def of Der of E(x)** **Admit L(x) is ln x and E(x) is e^x** 5/23
Last School Holiday # 7 R/C Pg 225 - 227 # 8 R/C pg 301 - 303 # 9 R/C pg 337 - 340 5 / 26	# 10 Pg 234 - 235 1 - 16 simplify 17,18 start with 1st item and end up with 2nd item. 19-22 Mumaugh Tables **Log properties** 5/ 27	# 11 Pg 242 1 - 16 Do each of 9 - 16 two different ways **No longer use "L" or "Log". Der of other exponential functions. Now use ln because we can "only" take der of ln** 5 / 28	**Important Lecture** # 12 Pg 246 1 – 6, 11 – 15 **Use integral notation method: What is it such that its der is the given ?** 5 / 29	**Important Lecture** # 13 Pg 251 1, 3, 7 - 17 odd **Point movement integrate acceleration to get velocity and integrate again to get location** 5 / 30
# 14 R/C pg 252 - 256 • • • • • • • • • • • • • • • • # 15 Pg 251 2 - 18 even (pblm 18 v = 3) Pg 260 1, 2, 4 - 6 6 / 2	# 16 Pg 257 1 – 9 **Important Lecture** **Natural Growth** **QUIZ in notebook** 6 / 3	#17 Pg 336 1 – 5 Pg 260 9 Q ? $\frac{a}{dt}$ 6 / 4	Put Last Year's Exam on the Boards 6 / 5	Exam Chapters 5, 6, 7 6 / 6
Last day to ask Questions 6 / 0	Final Exams Per 1, 2 6 / 10	Final Exams Per 3, 4 6/ 11	Final Exams Per 5, 6 6/ 12	No School ***except*** for those taking Make-Up Exams! 9:30 rm 32 6/ 13

Define $A(P, Q)$ to be the area trapped below the line $y = \frac{1}{x}$, above the x-axis between the lines $x = P$ and $x = Q$.



Let $P = 1$ $Q = 3$. use two rectangles to approximate the area.



$\sum$ each rectangle

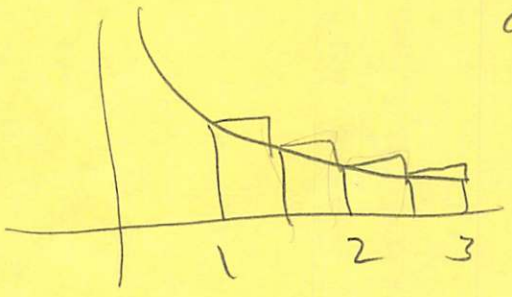
$1 \cdot 1 + \frac{1}{2} \cdot 1 =$

$1 + \frac{1}{2}$

$\frac{3}{2}$

use 4 rectangles

a better approximation:



$1 \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + \frac{2}{5} \cdot \frac{1}{2}$

$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{30+20+15+12}{60}$

$\frac{77}{60}$

take the limit as the widths of the rectangles goes to zero to get

$$A(P, Q) = \int_P^Q \frac{1}{x}\, dx$$

more

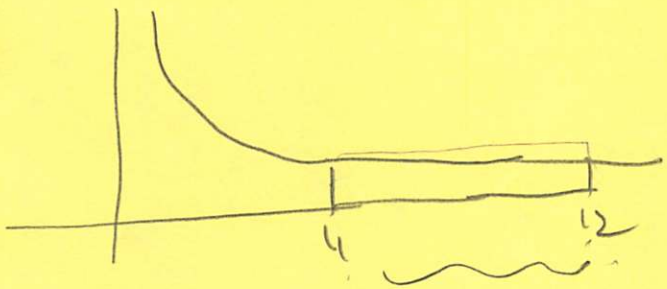
There are some interesting properties included in the functions A.

For example

$A(1,3) = A(4,12)$ because

$1 \cdot 2$

2



$\frac{1}{4} \cdot 8$

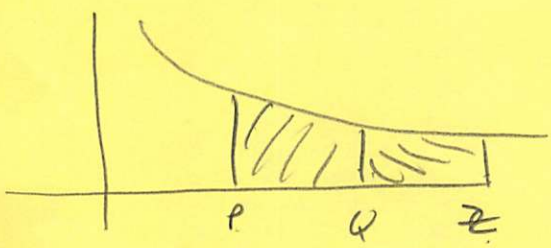
2

so $\frac{1}{3} = \frac{4}{12}$

$A(P,Q) = A(rP, rQ)$ where r is the common ratio.

Another property ...

$$A(P,Q) + A(Q,Z) = A(P,Z)$$



more

We define the function $L(x)$ as

$$L(x) = \begin{cases} -A(x,1) & \text{if } 0 < x < 1 \\ A(1,x) & \text{if } x \geq 1 \end{cases}$$

$L(x)$ is a special function.

One property of L is that

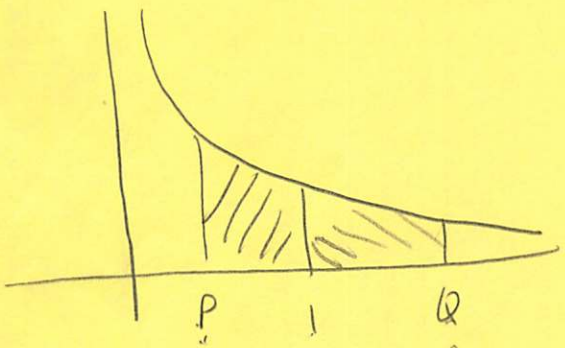
$$A(P,Q) = L(Q) - L(P) \text{ where } P < Q$$

There are 3 situations. $P < Q < 1$

$P < 1 < Q$.

$1 < P < Q$

I will show $P < 1 < Q$



$$A(PQ) = A(P,1) + A(1,Q)$$
$$= -L(P) + L(Q)$$
$$= L(Q) - L(P)$$

You will show the other two versions in HW.

These 3 more properties we will use:

$$L(xy) = L(x) + L(y)$$
$$L\left(\frac{y}{x}\right) = L(y) - L(x)$$
$$L(x^y) = y\,L(x)$$

These properties are really all the same.

$$L(x^2) = L(x \cdot x) = L(x) + L(x)$$
$$= 2L(x)$$

so $L(x^y) = L(\underbrace{x \cdot x \cdot x \cdots x}_{y \text{ terms}})$

$$= yL(x)$$

$$L\left(\frac{y}{x}\right) = L(x^{-1} \cdot y)$$
$$= L(x^{-1}) + L(y)$$
$$= -L(x) + L(y)$$
$$= L(y) - L(x)$$

Now "numerology tables"

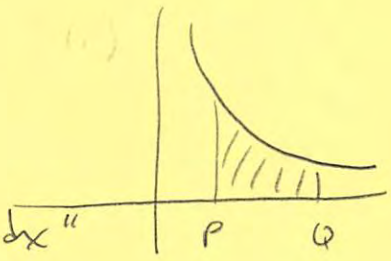
$L(1) = 0$
$L(2) = a$
$L(3) = b$
$L(4) = 2a$
$L(5) = c$
$L(6) = a+b$
etc.

until... everybody figures it out.

— END —

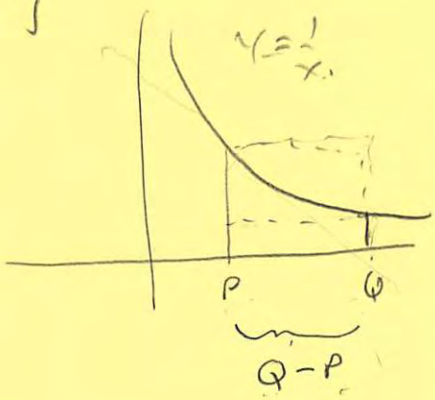
we have $L(x) = \begin{cases} -A(x,1) & \text{if } 0 < x < 1 \\ A(1,x) & \text{if } x \geq 1 \end{cases}$

where $A(P,Q) = \int_P^Q \frac{1}{x}\,dx$



This is translated to English as
"The integral from $x=P$ to $x=Q$ of $\frac{1}{x}\,dx$"

There are two approximating rectangles



Larger rectangle $= \frac{1}{P}(Q-P)$

Smaller rectangle $= \frac{1}{Q}(Q-P)$

So:

$$\frac{1}{Q}(Q-P) \leq A(P,Q) \leq \frac{1}{P}(Q-P)$$

$$\frac{1}{Q}(Q-P) \leq L(Q)-L(P) \leq \frac{1}{P}(Q-P)$$

$$\frac{1}{Q} \leq \frac{L(Q)-L(P)}{Q-P} \leq \frac{1}{P}$$

$$\lim_{Q \to P} \left(\quad \right)$$

$$\frac{1}{P} \leq \lim_{Q \to P} \frac{L(Q)-L(P)}{Q-P} \leq \frac{1}{P}$$

$$\therefore \lim_{Q \to P} \frac{L(Q)-L(P)}{Q-P} = \frac{1}{P}$$

$$L'(P) = \frac{1}{P}$$

Left side is def of der #1 of $L(x)$ at $x=p$.

thus $\left(L(x)\right)'\Big|_{x=p} = \frac{1}{p}$

Pg 2 of 2nd Lecture

the more generic formula

$$\frac{d\left(L(\star)\right)}{dx} = \frac{1}{\star} \cdot \star'$$

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what else do we know about $L(x)$?

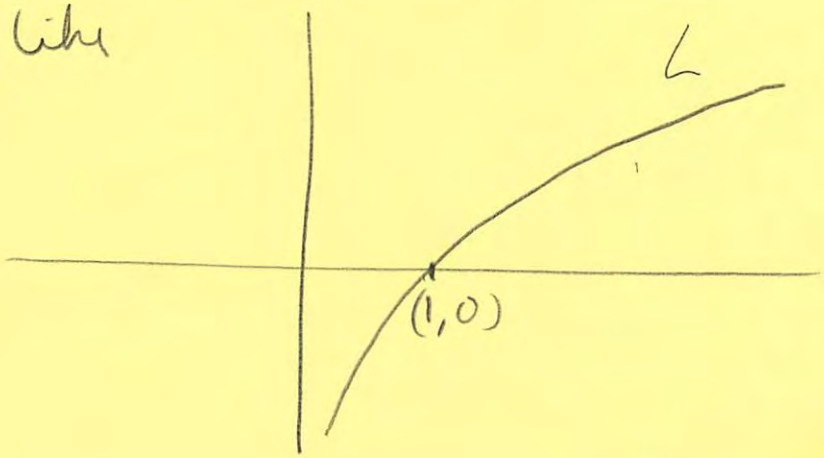
$x>0 \therefore L'(x) > 0$ so L is always increasing.

$L''(x) = \frac{-1}{x^2} < 0$ so L is concave down.

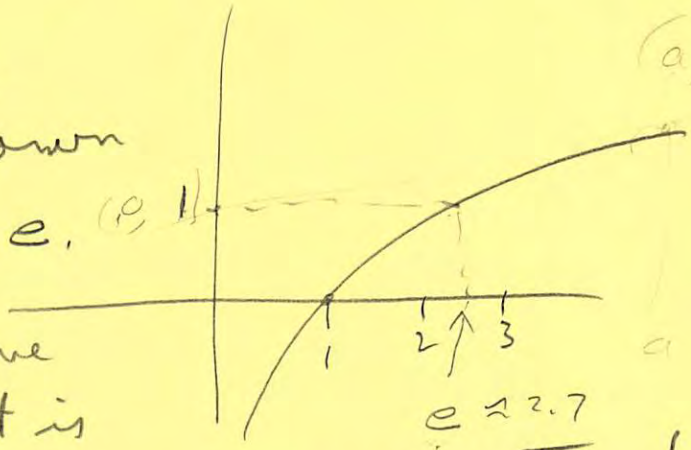
$L(1) = 0$ " $L\left(\frac{3}{3}\right) = L(3) - L(3)$

etc.

So L looks like



Since L is always increasing, it is reasonable that at some value of x, L(x) will equal 1.



This is an important x value with its own letter. Lower case e.

e's approximate value is 2.718 but it is like π in that it is a non repeating decimal.

So L(e) = 1 by definition

Problem: Find P such that the line tangent to the Graph of L at P passes through (0, 1).

We need slope and a point to get equation of a line.

We have the point (0, 1)

Assume P has coordinates (a, L(a)) so the the slope of any point on L is $\frac{1}{x}$

Slope at (a, L(a)) is $\frac{1}{a}$ = m

y-intercept 1, slope $\frac{1}{a}$ gives us

$$y = \frac{1}{a}x + 1$$

this is nice but we need the value of a so we can get the pentrodas coordinates $(a, L(a))$

$y = \frac{1}{a}x + 1$ passes through $a, L(a)$ so

$(e^2, 2)$

$$L(a) = \frac{1}{a} \cdot a + 1$$

$$L(a) = 1 + 1$$

★ $L(a) = 2$ so .... what is a?

Remember $L(e) = 1$

so $2L(e) = 2$

so $L(e^2) = 2$ comparing to ★

so $a = e^2$

the coordinate we need is $(e^2, 2)$

and the equation of the tangent line is

$$y = \frac{1}{e^2}x + 1$$

Open texts to pg 217

Lecture cont.

Oral ex

1)	$(1, ?)$	0	$m = 1$
2)	$(?, 1)$	e	$m = \frac{1}{e}$
3)	$(?, 0)$	1	$m = 1$
4)	$(e, ?)$	1	$m = \frac{1}{e}$
5)	$(e^4, ?)$	4	$m = \frac{1}{e^4}$
6)	$(?, 3)$	e^3	$m = \frac{1}{e^3}$

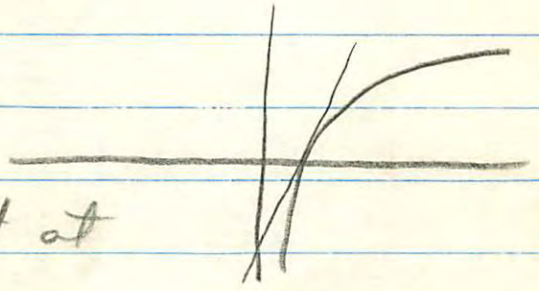
written. STILL lecture.

1. $L(x^2) = 2L(x) \therefore \frac{2}{x} = L'(x^2)$

2. $\frac{4}{x}$
3. $-\frac{1}{x}$
4. $-\frac{2}{x}$
5. $\frac{1}{2x}$
6. $\frac{1}{3x}$
7. $-\frac{1}{3x}$
8. $-\frac{2}{3x}$

9. find equations of line tangent to L at its x intercept.

Question: where is it tangent ie what are coordinate (1,0) x intercept and tangent at same time..



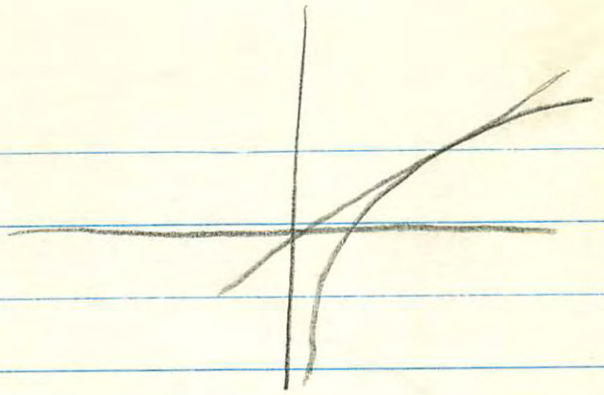
what is slope of tangent at (1,0)? $\frac{1}{1}$ is reciprocal of 1st derivative

$y - 0 = 1(x - 1)$

$y = x - 1$

END of Lecture

(14) $m = e$ tangent to L



Since $m = e$ then pt is

$\left(\frac{1}{e}, -1\right)$

$y + 1 = e\left(x - \frac{1}{e}\right)$

$y + 1 = ex - 1$

$y = ex - 2$

(15) $m = e^2$ $\left(\frac{1}{e^2}, -2\right)$

$y + 2 = e^2\left(x - \frac{1}{e^2}\right)$

$y + 2 = e^2x - 1$

$y = e^2x - 3$

(16) $m = \frac{1}{e}$ $(e, 1)$

$y - 1 = \frac{1}{e}(x - e)$

$y - 1 = \frac{1}{e}x - 1$

$y = \frac{1}{e}x$

(17) $m = \frac{1}{e^2}$ $(e^2, 2)$

$y - 2 = \frac{1}{e^2}(x - e^2)$

$y - 2 = \frac{1}{e^2}x - 1$

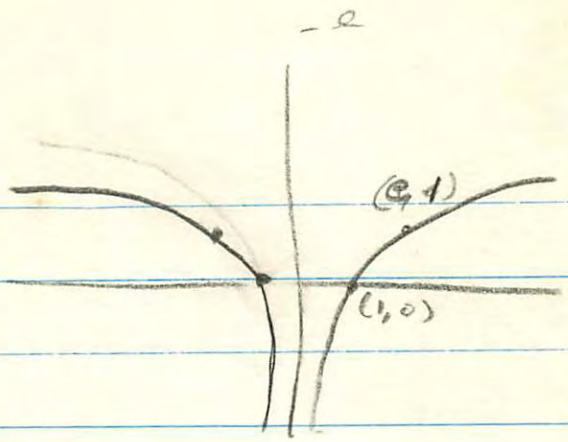
$y = \frac{1}{e^2}x + 1$

(18) Symmetric about $x=0$

$y = L(|x|)$

if $x > 0$ $\quad y = L(x)$

if $x < 0$ $\quad y = L(-x)$



(19) $f(x) = xL(x) \quad x > 0$

a) find $f'(x)$ $\quad u\,dv + u\,dv$

$u = x$

$du = 1$ $\quad L(x) + x \cdot \frac{1}{x}$

$v = L(x)$

$dv = \frac{1}{x}$ $\quad L(x) + 1$

b) find $f''(x)$

$$\frac{d(L(x)+1)}{dx} = \frac{1}{x}$$

c) $f' \overset{\text{set}}{=} 0 \overset{\text{set}}{=} L(x) + 1$

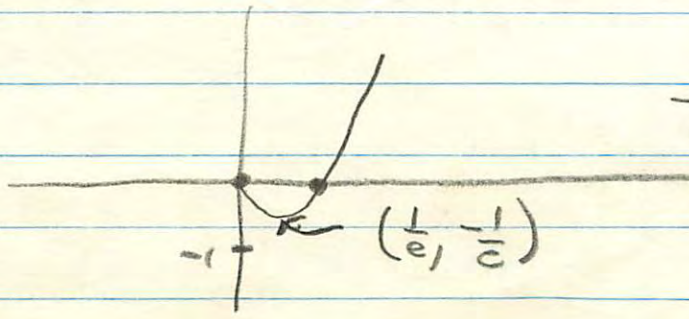
$L(x) = -1$ $\quad$ using $(?, -1)$ from $(x, L(x))$

$x = e^{-1}$

$f''\left(\frac{1}{e}\right) = e > 0 \therefore$ min pt of $f\left(\frac{1}{e}\right) = e^{-1}Le^{-1} = -e^{-1}L(e) = -e^{-1} \Rightarrow \left(\frac{1}{e}, -\frac{1}{e}\right)$

d) curve is concave downward always.

e)



x	$xL(x)$
0	0
1	$1 \cdot 0 = 0$

#20 $f(x) = \frac{L(x)}{x}$ $x>0$

a) $\frac{x\cdot\frac{1}{x} - 1\cdot L(x)}{x^2} = \frac{1-L(x)}{x^2}$

d) $f'(x)$

$u = L(x)$
$du = \frac{1}{x}$
$v = x$
$dv = 1$

b) f''

$u = 1 - L(x)$
$du = -\frac{1}{x}$
$v = x^2$
$dv = 2x$

$\frac{x^2\cdot\frac{-1}{x} - (1-L(x))2x}{x^4}$

$\frac{-x - 2x + 2xL(x)}{x^4}$

$\frac{-3x + 2xL(x)}{x^4}$

$\frac{-3 + 2L(x)}{x^3} \overset{set}{=} 0$

so $x \to 0$ we have $\lim \frac{1}{x}$

$\lim \left(\frac{-L(x)}{x}\right)$ $x\to 0$

PI at $x = e^{3/2}$

also $\lim_{x\to\infty} f(x) = 0$

and $\lim_{x\to 0} f(x) = -\infty$

c) $\frac{1-L(x)}{x^2} \overset{set}{=} 0$

$1 - L(x) = 0$
$L(x) = 1 \Rightarrow x = e$

$f(e) = \frac{1}{e}$

$f'(e) = \frac{-3+2\cdot 1}{e^3} = -\frac{1}{e^3}$

$f''(e) < 0$ $\therefore$ max, at $(e, \frac{1}{e})$

d) $f''(e) < 0$ $\therefore$ ~~Concave down~~ up.

Concave down $x < e^{3/2}$

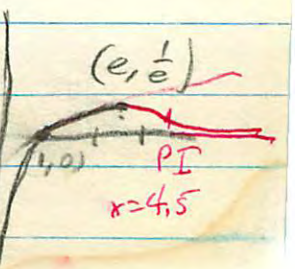
Concave up $x > e^{3/2}$

e)

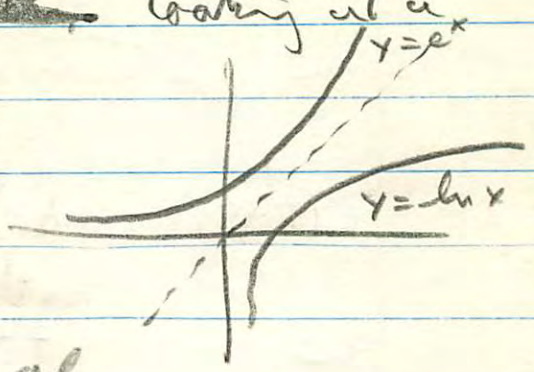
x	f(x)
1	0

$x>0$

only 1 extreme pt



There is a fctn called $L(x)$ ~~[illegible]~~ looking at a graph of the fctn $L(x)$ It would appear that L should have an inverse fctn id L^{-1}. That is the reflection of L across the line $y=x$, which is true. we call that fnctn E. generally it is written $y=e^x$ or $E=e^x$ all properties of Powers hold for E. Thm 4 pg 219 lists 2 properties of E.



Question what is E'? id what is $\frac{d(e^x)}{dx}$?

Given $Y=e^x$ operate using L
$L(Y)=L(e^x)$, $L(Y)=xL(e)$ Diff WRT x Implic.
$L(Y)=x$

$$\frac{1}{Y}\frac{dy}{dx}=1$$

$$\frac{dy}{dx}=Y \quad \text{but } Y=e^x \therefore$$

$$\frac{dy}{dx}=e^x \quad \text{that is to say } \frac{d(e^x)}{dx}=e^x.$$

e^x is its own derivative

$$\frac{d(e^u)}{dx}=e^u\frac{du}{dx}$$ important.

Quiz 5 Spt cal

Let $L(2)=a$ $L(3)=b$ $L(5)=c$ etc.

Determine, in terms of a, b, etc. $L(4\cdot 6)$

$2L2 + L3 + L2$

$L(125)$ $L(36)$ $L(24)$

5^3 $2(6^2 \cdot 2)$ $2(2^2 \cdot 6)$

$3C$ $2A+2B$ $2(L(2)+L(3))$

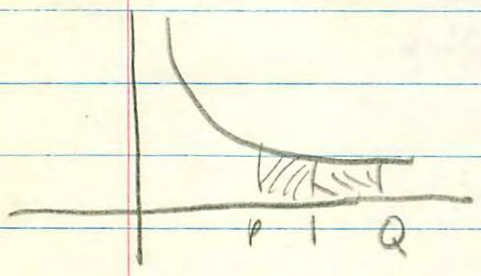
$3A+2B$

Given $L(x) = \begin{cases} A(1,x) & \text{if } x \geq 1 \\ -A(x,1) & \text{if } 0 < x < 1 \end{cases}$

and given $0 < P < 1 < Q$

Prove $A(P,Q) = L(Q) - L(P)$.

Draw a picture – include reasons for each step.



$A(P,Q) = A(P,1) + A(1,Q)$ A 2

$= -L(P) + L(Q)$ Def of L

$= L(Q) - L(P)$ order changed

Chapter 4

① a $\dfrac{(2ab^{-1})^{-2}}{(-a^2b)^{-3}} = \dfrac{(-a^2b)^3}{(2ab^{-1})^2} = \dfrac{-a^6b^3}{4a^2b^{-2}} = -\dfrac{a^4b^5}{4}$

b) $\dfrac{1}{\sqrt{a^{-2}+b^{-2}}} = \dfrac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}} = \dfrac{1}{\sqrt{\frac{b^2+a^2}{a^2b^2}}}$

$\dfrac{1}{\frac{\sqrt{b^2+a^2}}{ab}} = \dfrac{ab}{\sqrt{a^2+b^2}} = \dfrac{ab}{(a^2+b^2)^{\frac{1}{2}}}$

2) a $A(1,3) + A(3,6) = A(1,6) \quad x=6$

b) $A(1,3) = A(3,x) \quad x=9$

c) $A(x,2) = A(1,3) \quad x=\frac{2}{3}$

D) $\int_1^3 \frac{1}{t}\,dt = \int_2^x \frac{1}{t}\,dt$

$A(1,3) = A(2,x) \quad x=6$

3) a $L(2)=a \quad L(5)=b$

a) $L(10) = a+b$

b) $L(100) = L10^2 = 2L10 = 2(a+b)$

c) $L(25) = L5^2 = 2L5 = 2b$

D) $L(12.5) = L\frac{25}{2} = L5^2 - L2 = 2L5 - L2 =$
$2b-a$

3)

e) $A(2,5) = L5 - L2 = b - a$

f) $\int_2^5 \frac{1}{x}\,dx$ = Same as e

4) $f(x) = L\left(\frac{1}{x^2}\right) = L x^{-2} = -2 L x =$

$f'(x) = \frac{-2}{x}$

$f(x) = L\left(\frac{2}{x}\right) = L(2x^{-1}) = -L2x =$

$\log_b a = x \Rightarrow b^x = a$

$\log(ab) = \log a + \log b$

$\log \frac{a}{b} = \log a - \log$

$\log a^b = b \log a$

$f(f^{-1}(x)) = Y$

$f^{-1}(f(x)) = X$

$\log_b(b^x)$

$x \log_b b$

x

$b^{\log_b x} = Y$

$\log_b b^{\log_b x} = \log_b Y$

$\log_b x \log_b b = \log_b Y$

$\log_b x = \log_b Y$

$\therefore x = Y$

So $b^{\log_b x} = x$

$\log_b b = 1$

$b^{\log_b a} = a$ inverse operations.

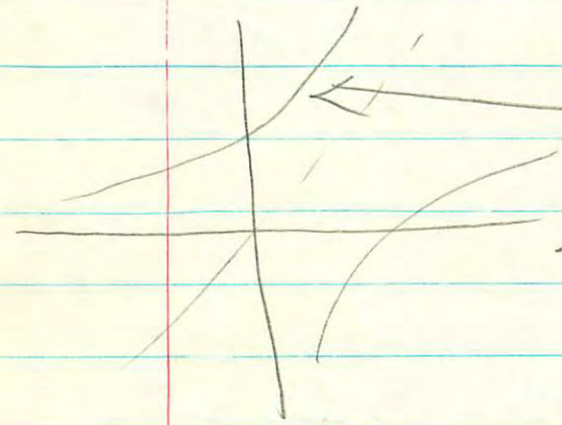
$\log_b a = \frac{\log_c a}{\log_c b}$

Hw pg 284 1-16 Simplify

17,18 Start with $x =$ 1st one get 2nd one

19-22 use "manual" tables

we should not be surprised that there is an inverse fcts because $y = \ln x$ passes both the vertical and horizontal line tests.



so $f(g(x)) = f(e^x) = \ln e^x = x \ln e = x$ as it should since $\ln x$ and e^x are.....

Let $g(f(x)) = Y$ $\Rightarrow$ $\ln x \cdot \ln e = \ln y$

$e^{\ln x}$ $\ln x \cdot 1 = \ln y$

$\therefore x = y$

$e^{\ln x} = Y$ but $Y = g(f(x))$

$\ln(\quad) = \ln(\quad)$ so $g(f(x)) = x$

$\therefore$ INVERSE fcts.

3rd Lecture 2:40-3:40

From now on, NO more: L(x), will be Pgl allowed we replace ANY Reference to L, L(x), Log with ln x

Natural Log: $\log_e x = \ln x$

ANY Reference to E(x) will be changed to e^x.

We don't even copy L or E from text book,

why ln x? because $\frac{d(\ln x)}{dx} = \frac{1}{x}$

we are NOT able to take a d/dx of $\text{Log}_{10} x$ as it stands.

$y = \text{Log}_{10} x = \frac{\ln x}{\ln 10}$, $y' = \frac{1}{\ln 10} \cdot \frac{1}{x}$

$= \frac{1}{\log_e 10}$

$= \frac{\log_{10} e}{x}$

Put it back in the form of the original

Now go to Pg3 then on to Pg3

Goto Pg3.

Also $y = e^x$

$y' = e^x$ we cannot take d/dx of any other exponential ie with different base

$y = 4^x$

$y' = ?$

$y = 4^x$

Version 1

$\ln y = \ln 4^x$

$= x \ln 4$

$\frac{1}{y}\frac{dy}{dx} = \ln 4$

$\frac{dy}{dx} = y \ln 4$

$= 4^x \ln 4$

Version 2

$y = e^{\ln 4^x}$

$= e^{x \ln 4}$

$y' = e^{x \ln 4}(\ln 4)$

$= 4^x \ln 4.$

$y = e^{e^x}$

$\ln y = \ln e^{e^x}$

$= e^x \ln e$

$= e^x$

$\frac{1}{y}\frac{dy}{dx} = e^x$

$\frac{dy}{dx} = e^{e^x} \cdot e^x$

$= e^{e^x + x}$

Now open text to Pg 242 Do odd ex.

Pg 3

$$y = \sqrt{\frac{\sin e^{3x}}{(1+2x^5)^3}}$$

$$\ln y = \ln \sqrt{\quad}$$

$$= \frac{1}{2}\ln \sin e^{3x} - \frac{3}{2}\ln(1+2x^5)$$

$$= \frac{3e^{3x}\cos e^{3x}}{2\sin e^{3x}} - \frac{3 \cdot 5 \cdot 2x}{2(1+2x^5)}$$

$$= \left(\frac{3e^{3x}\cos e^{3x}}{2\sin e^{3x}} - \frac{15x}{1+2x^5}\right)\sqrt{\frac{\sin e^{3x}}{\quad}}$$

If we don't use Log Properties first, we have a quotient rule to make a mess of things.

$$y = \log_5 \sqrt{\frac{3x^2}{(2x-1)^3}}$$

$$\frac{\ln \sqrt{\frac{3x^2}{(2x-1)^2}}}{\ln 5}$$

$$\frac{1}{\ln 5}\left[\frac{1}{2}\ln 3x^2 - 2\ln(2x-1)\right]$$

$$\frac{1}{\ln 5}\left[\frac{6x}{2 \cdot 3x^2} - \frac{2 \cdot 2}{2x-1}\right]$$

$$\frac{1}{\ln 5}\left[\frac{1}{x} - \frac{2}{2x-1}\right]$$

$$\frac{1}{\ln 5}\left[\frac{2x-1-2x}{x(2x-1)}\right]$$

$$\frac{-1}{\ln 5\,(x(2x-1))}$$

Lecture

examples

Given of Days of

~~HW # 6 assigned.~~

HW #9 pg 242

$y = \sqrt{\frac{5\sin e^{3x}}{(1+2x5)^3}}$

$y = 5^{-3x^2}$

$y = e^{\sqrt{\frac{8x+2}{(x-1)^5}}}$

~~$y = \frac{7x^5}{3}$~~

$y = e^{x^x}$

$y = e^{x^e}$

$y = \log_5 \sqrt{\frac{3x^2}{(2x-1)^3}}$

17 $\log_a b$, $\log_b a$ they are inverse

$\log_a b = \frac{\log b}{\log a}$ and $\log_b a = \frac{\log a}{\log b}$

18. $\log_{\frac{1}{a}} x$; $\log_{\frac{1}{a}} x$ they are Negatives.

$\frac{\log x}{\log a}$ $\frac{\log x}{\log \frac{1}{a}} = \frac{\log x}{\log a^{-1}} = \frac{\log x}{-\log a} = -\frac{\log x}{\log a}$

19. $\log_2 3 = \frac{\log 3}{\log 2}$ Skip 19-22

23, $3^x = 2^{\log_2 x}$ Skip 23-26

27. $z = \frac{a^x}{a^y}$: $\log\left(\frac{a^x}{a^y}\right) = \log a^x - \log a^y = \log z$

$x \log a - y \log a =$

$(x-y) \log a =$

$\log a^{(x-y)} = \log z$

$a^{x-y} = z$

(28) $(ab)^x = a^x b^x$

Let $(ab)^x = y$

$\log (ab)^x = \log y$

$x \log (ab)$

$x [\log a + \log b]$

$x \log a + x \log b$

$\log a^x + \log b^x$

$\log a^x b^x$

$a^x b^x = y = (ab)^x$

235

(29) $a^x = b^x$ Given

$\log_a a^x = \log_a b^x$

$x \log_a a = x \log_a b$

$\log_a a = \log_a b$

$a = b$

(30) $\log_a \frac{x}{y} = \log_a x - \log_a y$

$$\frac{L\left(\frac{x}{y}\right)}{L(a)} = \frac{L(x) - L(y)}{L a} = \frac{L(x)}{L(a)} - \frac{L(y)}{L(a)} = \log_a x - \log_a y$$

Using Thm. pg 210

$L\left(\frac{x}{y}\right) = L\left(x \cdot \frac{1}{y}\right) = L(x) + L\left(\frac{1}{y}\right)$

$L(x) + L(y^{-1})$

$L(x) - L(y)$

(31) Same Style as 30

The graph of the hyperbolic cosine (cosh) is called a catenary if a flexible cable suspended by its ends and hanging under its own weight such as an electric transmission line. The tension on the cable at any point may be computed in terms of hyperbolic fcns. An actual problem involving hanging cables involves material covered in second semester calculus. So this serves only as a minor introduction.

$$\cosh u = \tfrac{1}{2}(e^u + e^{-u})$$

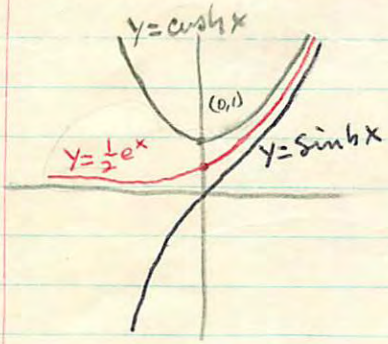
$$\sinh u = \tfrac{1}{2}(e^u - e^{-u})$$

The reason for this obviously close similarity to standard circular fcns is a result of the fact that many of the identities in the circular fcns have very close analogies in the hyperbolic fcns.

For example $\cos^2 x + \sin^2 x = 1$ and
$\cosh^2 x - \sinh^2 x = 1$

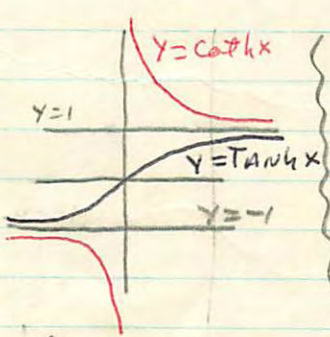
$\tanh u = \dfrac{\sinh u}{\cosh u}$ etc.

$\cosh(-x) = \cosh x$ and $\sinh(-x) = -\sinh(x)$

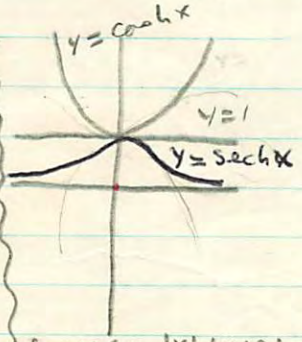


For x Large and >0
$\cosh x \approx \sinh x \approx \frac{1}{2}e^x$

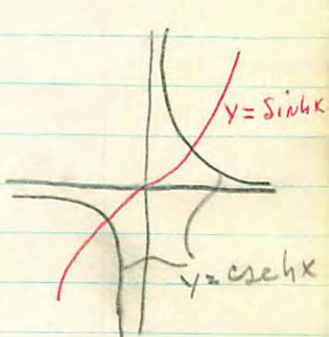
For x <0 |x| Large
$\cosh x \approx -\sinh x \approx \frac{1}{2}e^{-x}$



For x Large >0
$\tanh x \approx \coth x \approx 1$
For x<0 |x| Large
$\tanh x \approx \coth x \approx -1$



For x<0 |x| Large
$\operatorname{sech} x \approx -\operatorname{csch} x \approx 2e^x \approx 0$
and,
For x Large and >0
$\operatorname{sech} x \approx \operatorname{csch} x \approx 2e^{-x} = 0$



Proof $\log_a e = \frac{1}{\log a}$

$\log_a e = \frac{Le}{La} = \frac{1}{L(a)} = \frac{1}{\log a}$ (See pg 233 for review of Last step.)

Proof

$$\frac{d(\log_a x)}{dx} = \log_a e \cdot \frac{1}{x}$$

$$\frac{d(\log_a x)}{dx} = \frac{d\left(\frac{L(x)}{L(a)}\right)}{dx} = \frac{1}{L(a)} \frac{d(L(x))}{dx} =$$

$$\frac{1}{L(a)} \cdot \frac{1}{x} = \log_a e \cdot \frac{1}{x}$$

find $\frac{d \log_{10} x}{dx} = \log_{10} e \cdot \frac{1}{x} = \frac{0.434}{x}$

Proof

$$\frac{d\, a^x}{dx} = \log a \cdot a^x$$

we define $\log x \equiv \ln x$.

If we mean common log we will say $\log_{10} x$.

Thm 3 $\frac{d}{dx}\text{Log} f(x) = \frac{f'(x)}{f(x)}$ $\qquad \frac{d\,\text{Log}\,u}{dx} = \frac{1}{u}\frac{du}{dx}$

Proof

$\frac{d\,\text{Log} f(x)}{dx} = \frac{d\,\text{Log}\,u}{du} \cdot \frac{du}{dx}$ Chain Rule.

Let $u = f(x)$ $\qquad \frac{d(\text{Log}\,u)}{du} = \frac{1}{u} = \frac{1}{f(x)}$

$\frac{du}{dx} = f'(x)$

$\frac{d\,\text{Log}\,u}{du} \cdot \frac{du}{dx} = f'(x)\frac{1}{f(x)} = \frac{f'(x)}{f(x)}$

Summary

Thm 4 $\frac{d}{dx}e^{f(x)} = e^{f(x)} \cdot f'(x)$ $\qquad \frac{de^u}{dx} = e^u \frac{du}{dx}$

Let $u = f(x)$ $\qquad \frac{de^{f(x)}}{dx} = \frac{de^u}{du} \cdot \frac{du}{dx}$

$du = f'(x)dx$

$\frac{d(e^u)}{du} = e^u$ $\qquad e^u \cdot f'(x) \Rightarrow e^{f(x)} f'(x)$

Eg 2 pg 241

a) $\frac{de^{kx}}{dx} = e^{kx}\frac{d(kx)}{dx} = e^{kx} \cdot k = ke^{kx}$

b) $\frac{d\,\text{Log}\sqrt{x^2+1}}{dx} = \frac{d\left(\text{Log}\,(x^2+1)^{\frac{1}{2}}\right)}{dx} = \frac{d\left(\frac{1}{2}\text{Log}\,(x^2+1)\right)}{dx} =$

$\frac{1}{2(x^2+1)} \cdot \frac{d(x^2+1)}{dx} = \frac{1}{2(x^2+1)} \cdot 2x = \frac{x}{x^2+1}$

Hw pg 242 9-12 a d

Hw pg 242-243 13-18

order:

① up
② down
③ up
④ down
⑤ upward
⑥ upward
⑦ downward
⑧ +
⑨ −

all logarithmic fctns pass through the point (1,0) because

$\log_a x = y \Rightarrow a^y = x$ note a is any base.

all exponential fctns pass through

(0,1) $y = a^x$ Since $a > 0$ raised to zero power is one.

$$\frac{d \log u}{dx} = \frac{1}{u}\frac{du}{dx}$$

(9) $f(x) = \log \sqrt[3]{x^2+1}$

$$f'(x) = \frac{d \log \sqrt[3]{x^2+1}}{dx} = \frac{\frac{1}{3} d(\log (x^2+1))}{dx} = \frac{2x}{3(x^2+1)}$$

(10) $f(x) = \log\left(\frac{1}{x^2+1}\right)$

$$-d \log (x^2+1) = \frac{-2x}{(x^2+1)}$$

(11) $f(x) = e^{x^2}$ $\quad f'(x) = 2x e^{x^2}$

(12) $f(x) = e^{-x^2}$ $\quad f'(x) = -2x e^{-2x}$

(13) $f(x) = e^{1-x}$ $\quad f'(x) = -e^{1-x}$

(14) $f(x) = e^{2x-1}$ $\quad f'(x) = 2e^{2x-1}$

(15) $f(x) = \log x\sqrt{x^2+1}$

$$\frac{dR}{dx} = \frac{4x^3+2x}{2\sqrt{x^4+x^2}} = \frac{4x^3+2x}{2x\sqrt{x^2+1}}$$

$$\frac{dP}{dx} = \frac{4x^2+2}{2\sqrt{x^2+1}} = \frac{2x^2+1}{\sqrt{x^2+1}}$$

$f'(u) = \frac{1}{u}\frac{du}{dx}$

$$\frac{2x^2+1}{x\sqrt{x^2+1}\ \sqrt{x^2+1}} = \frac{2x^2+1}{x(x^2+1)}$$

$R = x\sqrt{x^2+1}$

$R = \sqrt{x^4+x^2} = u^{\frac{1}{2}}$

$u = x^4+x^2$

$\frac{du}{dx} = 4x^3+2x$

$\frac{dP}{du} = \frac{1}{2P} = \frac{1}{2\sqrt{x^4+x^2}}$

OR

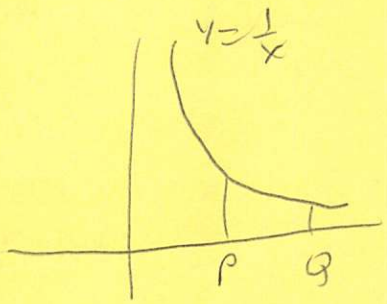
$\log x + \log\sqrt{x^2+1}$ etc.

$\log x + \frac{1}{2}\log x^2+1$

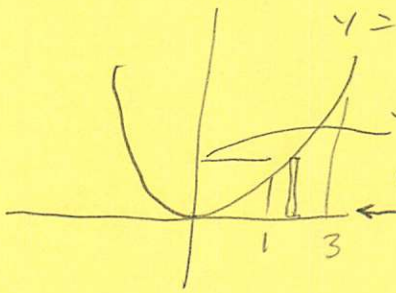
$$\frac{1}{x} + \frac{2x}{2(x^2+1)} = \frac{x^2+1+x^2}{x^2+1} = \frac{2x^2+1}{x^2+1} = f'(x)$$

$\int x^2\,dx$ translates to "what is it that when you take the derivative you get x^2?"

we defined $\int_P^Q \frac{1}{x}\,dx = L(Q) - L(P)$ a week ago or so



we can actually can find the area under almost any curve.



This would be written $\int_1^3 x^2\,dx$

x^2 because of the fctn

dx because we could "sweep out" the area with a "rectangle" moving along the x axis.

"what is it that when we take the der of it we get x^2?"

$\frac{1}{3}x^3$

So $\int_1^3 x^2\,dx = \frac{1}{3}x^3\Big|_1^3 = \frac{1}{3}(27) - \frac{1}{3}(1)$

definite integral has limits

$9 - \frac{1}{3}$

$\frac{26}{3}$

$\int x^2\,dx = \frac{1}{3}x^3 + C$ because the der of $\left(\frac{1}{3}x^3 + C\right)$ is x^2 no matter what C is.

$\int x^N\,dx = ?$ $\quad \frac{x^{N+1}}{N+1} + C$?

well, almost when doesn't it work?
when $N = -1$

so $\int x^N\,dx = \begin{cases} \frac{x^{N+1}}{N+1} + C & \text{if } N \neq -1 \\ ? & \text{if } N = -1 \end{cases}$

if $N = -1$ then

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

why $|x|$?

$$y = \ln|x| = \begin{cases} \ln(-x) & \text{if } x < 0 \\ \ln x & \text{if } x > 0 \end{cases}$$

$\frac{1}{(-x)}(-1) = \frac{1}{x}$

Pg 243-246

Antiderivatives or indfinite integrals

Given $d(F) = f(x)$ and $dg(x) = h(x)$.

F is the integral of f. That is to say, $d(F) = f$. Generally this is the easiest method to use to find integrals. Remember what you have already done.

eg #1, find $\int x^2 dx = \frac{1}{3}x^3$

$$\int ke^{kx} dx = e^{kx}$$

$$\int e^{kx} dx = \frac{1}{k} e^{kx}$$

$$\int a^x dx = (\log a) a^x \quad (\text{Pg } 240)$$

eg #2 find

A) $\frac{d\left(\text{Log } \sqrt{(x^2+1)}\right)}{dx} = \frac{d\, \frac{1}{2}\text{Log}(x^2+1)}{dx} = \frac{x}{x^2+1}$

B) $d\left(\log \frac{\sqrt{x^2+1}}{3}\right) \frac{d(\text{Log } \sqrt{x^2+1} + \text{Log } (3))}{dx} =$

$\frac{d(\log \sqrt{x^2+1})}{dx} = \frac{x}{x^2+1}$ since Log 3 is constant.

It should seem odd that two different fcts have the same derivative.

c) $\frac{d(x^2+3x+5)}{dx} = 2x+3$

d) $d(x^2+3x+11) = 2x+3$

Notice the difference is only in a constant being added or subtracted from the basic fctn.

e) $\frac{d(x^2+3x+k)}{dx} = 2x+3$

When looking for integrals we have what is called "constant of integration".

$$\int 2x+3 \, dx = x^2+3x+c$$

Polynomial fcns are especially nice since we can usually determine the integral with ease.

$$\int x^a dx = \frac{x^{a+1}}{a+1} + c \quad \text{if } a \neq -1$$

If $a = -1$ we have a minor problem.

$\int x^{-1} dx = \int \frac{1}{x} dx$. Our problem is easily solved since we remember $\frac{d(\ln|x|)}{dx} = \frac{1}{x}$

$\therefore \int \frac{1}{x} dx = \ln|x| + c$.

we must use $|x|$ rather than x however.

Log is defined only for positive values.
Thus if $x < 0$, $-x > 0$

So $\log(-x)$ is defined if $x < 0$

using $\frac{d(\log u)}{du} = \frac{1}{u}\frac{du}{dx} \Rightarrow \frac{d(\log(-x))}{dx} = \frac{1}{-x} \cdot -1 = \frac{-1}{-x} = \frac{1}{x}$

clearly both $\log x$ and $\log(-x)$ are possible thus here we use $\log|x| + c$ as the integral

orales

① $\frac{x^2}{2}+c$ ② $\frac{x^4}{4}+c$ ③ $Lg|x|+c$ ④ e^x+c

⑤ $\frac{1}{2}e^{2x}+c$ ⑥ $-e^{-x}+c$ ⑦ $2Lg|x|+c$

⑧ $\frac{2x^{3/2}}{3}+c$ ⑨ $\frac{2}{3}x^{\frac{3}{2}}+c$

escritos.

① $\frac{x^4}{4}+\frac{x^3}{3}+\frac{x^2}{2}+x+c$

② $x^4+x^3+x^2+x+c$

③ $\frac{x^2}{2}+Lg|x|+c$

④ $\frac{x^3}{3}+Lg|x|+c$

⑤ $x+Log|x|-\frac{1}{x}+c$

⑥ $\frac{x^4}{4}+\frac{x^2}{2}+Lg|x|-\frac{1}{2x^2}+c$.

⑦ $(Log_2 e)2^x+c$

⑧ $(Log_{10} e)10^x+c$

⑨ $-\cosh x+c$

⑩ $\sin x+c$

⑪ $f(x)=xe^x-x$

$f'(x)=xe^x+e^x-1$ $\therefore$ $\int(xe^x+e^x-1)dx=xe^x-x+c$

$$g(x) = \frac{1}{2}\ln\left(\frac{x-1}{x+1}\right)$$

$$x = \frac{1}{2}\ln\left(\frac{z-1}{z+1}\right)$$

$$e^{2x} = \frac{z-1}{z+1}$$

$$e^{2x}z + e^{2x} = z - 1$$

$$(e^{2}x - 1)z = -(e^{2}x + 1)$$

$$z = \frac{-(e^{2x}+1)}{e^{2x}-1}$$

$$e^{2x} - 1 \overset{\text{Set}}{=} 0$$

$$e^{2x} = 1$$

$$x = 0$$

(12) $g(x) = x\log x - x$

$g'(x) = 1 + \log x - 1 = \log x$

$\int \log x\,dx = x\log x - 1$

(3) $g(x) = \frac{1}{2}\log\frac{x-1}{x+1}$ = method is POOR.

if $x-1<0$ and $x+1<0$ then $\frac{1}{2}[\log(x-1) - \log(x+1)]$ is NOT defined.

$g'(x) = \frac{1}{2}\left[\log(x-1) - \log(x+1)\right]$

$g'(x) = \frac{1}{2}\left[\frac{1}{x-1} - \frac{1}{x+1}\right]$

$= \frac{1}{2}\left[\frac{x+1-x+1}{x^2-1}\right] = \frac{1}{2}\left[\frac{2}{x^2-1}\right] = \frac{1}{x^2-1}$

this method is better

$\int \frac{1}{x^2-1}\,dx = \frac{1}{2}\log\left(\frac{x-1}{x+1}\right) + c$

(13) $g(x) = \frac{1}{2}\log\frac{x-1}{x+1}$ $\quad g'(x) = \frac{x+1}{2(x-1)}\cdot\frac{d\left(\frac{x-1}{x+1}\right)}{dx}$

also discuss inverse of $g(x)$ then $R \neq 0$.

$u = x-1$ $\quad \frac{x+1-(x-1)}{(x+1)^2}$

$du = 1$

$v = x+1$

$dv = 1$

$\frac{x+1-x+1}{(x+1)^2} = \frac{2}{(x+1)^2}$ $\quad \therefore g'(x) = \frac{x+1}{2(x-1)}\cdot\frac{2}{(x+1)^2} =$

$g'(x) = \frac{1}{x^2-1}$

(15) $g(x) = \frac{1}{2}\log\left(\frac{x-1}{x+1}\right)$ $\quad$ Case I $x-1>0$, $x>1$; $x+1>0$, $x>-1$; $\therefore x>1$ $\quad$ Case II $x-1<0$, $x<1$; $x+1<0$, $x<-1$; $\therefore x<-1$

$\text{Dom} \Rightarrow x>1$ or $x<-1$

Notice Problem if

Example: "Find the function whose graph has the following properties:

(1) its slope at each point is twice the abscissa of the point.

(2) the graph passes through $(2,1)$.

Solution

translate: $\frac{dy}{dx} = 2x$

Separate variables: $dy = 2x\,dx$

integrate

$$\int dy = \int 2x\,dx$$

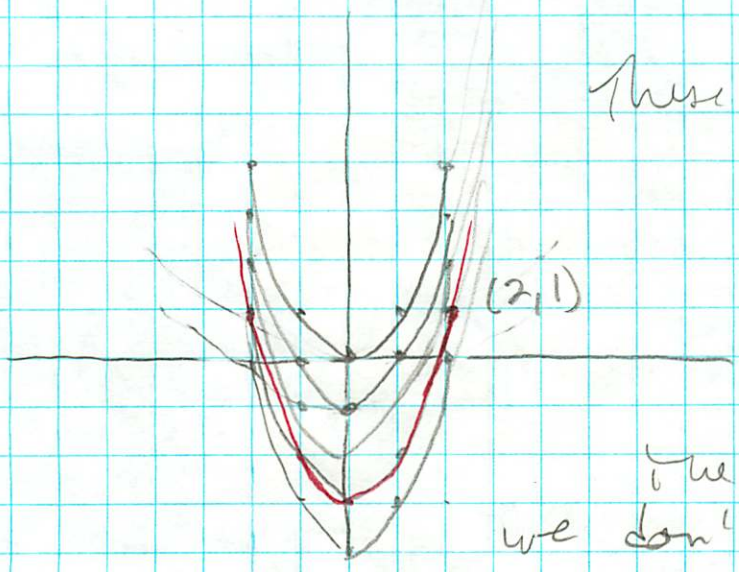
what is it, when I take the derivative I get dy? $y + c_1$

what is it, when I take the derivative I get $2x$? $\frac{1}{2}x^2 + c_2$

$$y + c_1 = x^2 + c_2 \qquad \text{let } c_2 - c_1 = c$$

$$y = x^2 + c$$

Now "which of the family of curves do we want?



These are all part of the

$y = x^2 + c$

family.

only the red curve satisfies the requirement.

we don't graph to solve however.

$y = x^2 + c$ let $x=2$, $y=1$

$1 = 2^2 + c$

$c = -3$

so the particular equation is $y = x^2 - 3$ which satisfies the situation that every when the slope is twice the x value of the point AND the graph passes through (2, 1)

The "derivative" of something is
"the rate of change" of that something.

we used that definition in the last chapter.

consider s to be distance in terms of time t.

what would be the rate of change of distance?
speed or velocity.

are speed and velocity the same thing?
No velocity contains information about direction.
speedometer reading is speed.

So $(s(t))' = v(t)$

what would be the rate of change of velocity?
acceleration. How fast velocity is changing.

So $(v(t))' = a(t)$

another way to think of it is

$\int a(t)\,dt = v(t)$ and

$\int v(t)\,dt = s(t)$.

Is it possible for two cars to be side by side on the freeway with speedometer readings of exactly 60 mph to have different accelerations?

Yes one could be slowing to 60 mph while the other is increasing to 60 mph. Negative acceleration is called deceleration.

Example.

A point is moving on the s-axis in such a way that when its position (measured in feet) is s, its velocity (measured in ft/sec) is $-2s$. When $t=0$ the point s is at 10. (This is called "initial conditions")

Find ① the general position function for the distance from zero ($s(t)$)

② when will the point be 5 units from zero?

Solution

Translate $\frac{dS}{dt} = -2S$

Separate variables $\frac{dS}{S} = -2\,dt$

Integrate $\int \frac{dS}{S} = \int -2\,dt$

$\ln|S| = -2t + C_1$

For a moment, assume S is positive. $\ln S = -2t + C_1$

raise e to power of each side $e^{\ln S} = e^{-2t+C_1}$

In order to solve for S $S = e^{-2t} \cdot e^{C}$

what is e^C? a constant. call it c $S = Ce^{-2t}$

Now.... If $S < 0$ we could make Ce^{-2t} negative by just making C negative. so we haven't compromised our function.

↓

So far we have $S = ce^{-2t}$

but we know an initial condition.

when $t = 0$ $S = 10$

so $10 = ce^{-2(0)}$

$\therefore c = 10$

So the particular solution is

$$S = 10e^{-2t}$$

Does this satisfy the requirements as outlined in the problem?

$S(0) = 10$? yes.

What about $\frac{dS}{dt}$?

what is $\frac{dS}{dt}$?

$$\frac{dS}{dt} = -2(10e^{-2t})$$

but $10e^{-2t}$ is S so

$$\frac{dS}{dt} = -2S$$

as it should be.

— END —

Quiz Between Pg 246 odds or Pg 246 Exercises 1, 2 — 15 minutes

Given y, find y'

(a) $\ln \sqrt[3]{\frac{2x+1}{2x-1}}$

$\frac{1}{3}\left\{\ln(2x+1) - \frac{1}{3}\ln(2x-1)\right\}$

$\frac{1}{3}\left\{\frac{2}{2x+1} - \frac{2}{2x-1}\right\} = \frac{-2}{4x^2-1}$

$4x-2-4x-2$

b) $y = 3^x$ $\quad e^{x\ln 3}$

$3^x \ln 3$

c) $y = e^{e^x}$ $\quad \ln y = e^x$ $\quad e^{e^x}\cdot e^x$

$\frac{1}{y}\frac{dy}{dx} = e^x$

d) $\log_3 e^x$ $\quad x \log_3 e \Rightarrow \log_3 e$

$x \frac{\ln e}{\ln 3} \rightarrow \frac{1}{\ln 3}$

e) $\log_7 x^x$ $\quad x \frac{\ln x}{\ln 7}$ $\quad \frac{1}{\ln 7}\{\ln x + 1\} =$

$\cdot \log_7 e\{\ln x + 1\}$

name _____

6 - 10 + ✓
5 0
4 −
3 =
0, 1, 2

1) Given $y = a^b$. Determine $y'(x)$ for every possibility of a and b.

For example, a may or may not be a function of x. The same is true of b.
You need to cover every possible situation clearly.

(2)
If (1) a, b are constants $y' = 0$ or if $a = 1$ or $b = 0$
If a variable b is constant $y' = b a^{b-1}$ (3)
If a is constant b is variable $y = a^{\star}$ (4)

$\ln y = \star \ln a$

$\frac{1}{y}\frac{dy}{dx} = \ln a \, \frac{d\star}{dx}$

$\frac{dy}{dx} = a^b (\ln a) \frac{db}{dx}$

If both are variables

$y = a^b$ (5)

$\ln y = b \ln a$

$\frac{1}{y}\frac{dy}{dx} = \ln a \frac{db}{dx} + \frac{b}{a}\frac{da}{dx}$

$\frac{dy}{dx} = a^b \left\{ \ln a \frac{db}{dx} + \frac{b}{a}\frac{da}{dx} \right\}$

2) Given $y = \log_b a$. Determine $y'(x)$ for every possibility of a and b. (as above)

If $b = 1$, $b < 0$ (6) DNE

If a and b are constants (7) $y' = 0$

If $b = e$ a is fctn (8) $y' = \frac{1}{a}\frac{d\star}{dx}$

If $b \neq e$ a is fctn. (9) $y = \frac{\ln a}{\ln b}$

If b is fctn a is constant (10)

$y = \frac{\ln a}{\ln b} = (\ln a)(\ln b)^{-1}$

$y' = -(\ln a)(\ln b)^{-2}\frac{1}{b}\frac{db}{dx} = -\frac{\ln a}{b(\ln b)^2}\frac{db}{dx}$

$y' = \frac{1}{\ln b}\frac{1}{a}\frac{d\star}{dx}$

10 minute quiz

Exponential Quiz

17, 18 +
15, 16 ✓
13, 14 0
9–12 −
< 12 ?

Name _____
Period _____

Quiz score _____

Given: c is a constant.
e is the base of the natural log.
$\star$ is a function of x.

Find y'

1) $y = c^c$ — 0

2) $y = e^c$ — 0

3) $y = c^\star$ — $c^\star \ln c \, d\star$

4) $y = e^\star$ — $e^\star \, d\star$

5) $y = e^e$ — 0

6) $y = \star^c$ — $c\star^{c-1} d\star$

7) $y = \star^e$ — $e\star^{e-1} d\star$

8) $y = \star^\star = e^{\star \ln \star}$ — $\star^\star (\ln\star + 1) d\star$

9) $y = \log_c c$ — 0

10) $y = \log_e c$ — 0

11) $y = \log_c \star$ — $\frac{\ln\star}{\ln c}$ — $\frac{1}{\star \ln c} d\star$

12) $y = \log_e \star = \ln\star$ — $\frac{1}{\star} d\star$

13) $y = \log_c e$ — 0

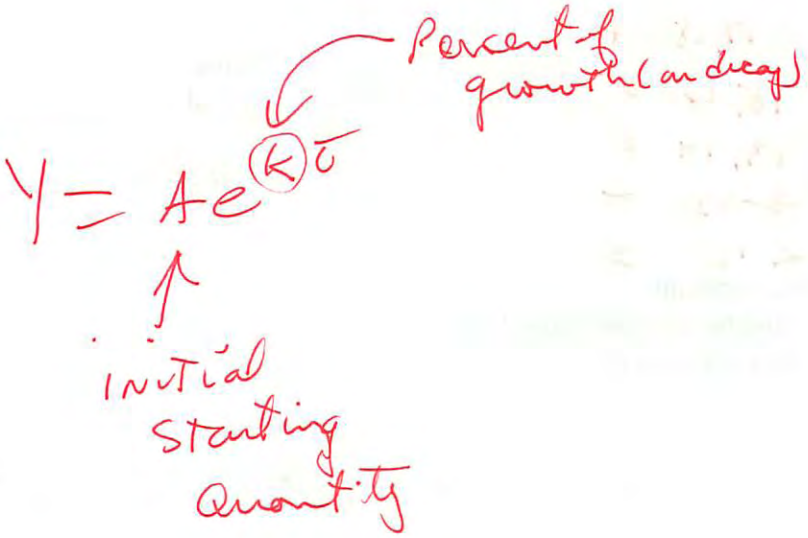
14) $y = \log_e e$ — 0

15) $y = \log_\star c$ — $\frac{\ln c}{\ln\star}$ — $\frac{-\ln c}{\star(\ln\star)^2} d\star$

16) $y = \log_\star e$ — $\frac{-1}{\star(\ln\star)^2} d\star$

17) $y = \log_\star \star$ — $= 1$ — $y' = 0$

18) $y = \log_\star \star^\star$ — $\star \log_\star \star = \star$ — $y' = d\star$


Percent of growth (or decay)
$y = Ae^{kt}$
INITIAL starting quantity

c is "a constant"
e is base of Natural Logs
$\star$ is a fcn of x

find y'

$y = c^c$ $y = e^c$ $y = c^\star$ $y = e^\star$ ~~$y = \star^e$~~

$y = e^e$ $y = \star^c$ $y = \star^e$ $y = \star^\star$

find y'

$y = \log_c c^c$ $y = \log_e c$ $y = \log_c \star$ $y = \log_e \star$

$y = \log_c e^e$ $y = \log_e e$ $y = \log_\star c$ $y = \log_\star e^e$

$y = \log_\star \star^\star$ $y = \log_\star \star^\star$

let students have 5 minutes on each section then
share discuss answers.
next I shared "differential equations"
a more specific bc of growth tomorrow.

Pg 247-251

Eg #1 "find the fctn whose graph has the following properties: ① its slope at each point is twice the abscissa of the point, and ② it passes through (2, 1).

From ① the slope is $\frac{dy}{dx} = 2x$ and ② when $x=2$ $y=1$.

we are asked to determine the fctn y that is the integral of $\frac{dy}{dx} = 2x$

$$\frac{dy}{dx} = 2x$$

$$dy = 2x\,dx$$

$$\int dy = \int 2x\,dx$$

$$y = x^2 + c$$

now using ② (2, 1) we determine c.

horizontal spring on frictionless surface.

$$1 = 2^2 + c$$

$$1 = 4 + c$$

$$c = -3$$

∴ fctn which satisfies given conditions is: $y = x^2 - 3$.

Verify it does $\frac{dy}{dx} = 2x$ as required? Yes.

description of motion of a spring

Eg #2 (Pg 248) A point is moving on a line, s-axis, in such a way that when its position (measured in feet) is s, its velocity measured in feet is $-2s$. when $t=0$ the position of the point is $s=10$. Find ⓐ the position of the point at time t, that is, the fctn s which describes position and ⓑ at what time is the particle 5 units from 0.

Velocity $\Rightarrow \frac{ds}{dt} = -2s$.

we need the fctn s which describes the position of P at some time t.

$$\frac{ds}{dt} = -2s \Rightarrow \int \frac{ds}{s} = -2\int dt \Rightarrow \text{Log}|s| + c_1 = -2t + c_2$$

Notice we have 2 constants this would become

$\text{Log}|s| = -2t + c_3$ where c_3 is $c_2 - c_1$

① the point is, we have a constant on the end and exactly which one it is makes No difference since we only imply that there is indeed a constant of integration included.

② Do we need $|s|$? No since we know s is a distance only, s must be Non-Negative.

result:

$\text{Log } s = -2t + c_3$ STILL we are interested in the fctn s not the log of s so operate w/ e.

★ $s = e^{-2t+c_3}$

Notice $e^{-2t+c} = e^{-2t} \cdot e^{c_3}$ thus e^c is another constant so we simplify ★ by:

$s = ce^{-2t}$ where $c = e^c$ (2 different c's but they are used only to represent (again) the constant of integration.

Now we are ready to determine exactly what that elusive value "c" actually is.

$s = 10$; $t = 0 \Rightarrow 10 = ce^0 \Rightarrow c = 10$ This is the "last c" we employed, the others are of No consequence. The fctn becomes $s = 10e^{-2t}$.

Next Question was when is the particle at a distance of 5 units ($s = 5$) from 0.

$s = 10e^{-2\tau}$ ←

$5 = 10e^{-2\tau}$

$\frac{1}{2} = e^{-2\tau}$

$\log(\frac{1}{2}) = -2\tau$

$-\log 2 = -2\tau$

$\frac{1}{2}\log 2 = \tau$ Since $\log 2 = .693$

$\frac{\log 2}{2} = .347$ seconds. $= \overline{\tau}$.

Question: With all our shifting of "c" earlier, did we actually derive the proper form?

method! See if $\frac{ds}{d\tau} = -2s$ as was originally stated. ~~TA the most general solution~~!

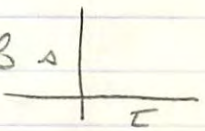
$s = 10e^{-2\tau}$ Take derivative WRT τ.

$\frac{ds}{d\tau} = -2(10)e^{-2\tau}$

$-2s$

~~Remember we said $s = ce^{-2\tau}$ so:~~

$\frac{ds}{d\tau} = -2s$, Notice: the actual value of s c made No difference!



Pg 249 the text describes laws of motion of a point moving on a vertical line under the influence of gravity. Natrl the acceleration of the point (actually "deceleration") is $-g$. We shall let time start at 0 ie $\tau = 0$ with position s_0 and ..., having velocity v_0. We shall have all derivatives WRT τ. Thus:

$s' = v$; $s'' = v' = a$

so Since acceleration is $-g$ we have:

$a = -g$ that is to say $v' = -g$

ie $\frac{d(v)}{dT} = -g \Rightarrow dv = -g\,dT$.

Take integral of both sides.

$v = -gT + c$

we have some information about v.

at time zero ($t = 0$) $v = v_0$ ∴

$v_0 = -g(0) + c \Rightarrow v_0 = c$. which yields

$v = -gT + v_0 = s'$ again taking the integral

$s = -\frac{1}{2}gT^2 + v_0 T + c$ again we know some specific conditions $T = 0$, $s = s_0$

$s = -\frac{1}{2}gT^2 + v_0 T + s_0$

$= -16gT^2 + v_0 T + s_0$

turn to page 69 compare the above fctn with that of the object being projected from the tower.

turn back to pg 249 + read sub script at bottom of page.

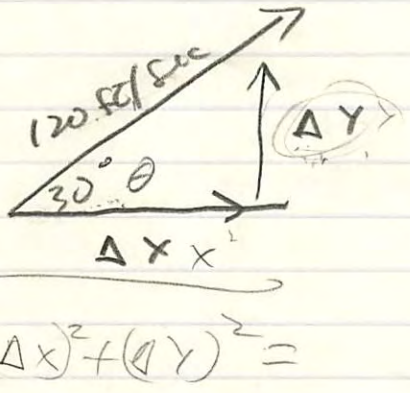
Compare $g = 32$ look for definitions of initial conditions on page 69.

The only place we seem to disagree is at v_0. Why is there a discrepancy? Our problem discusses vertical movement only while the problem on pg 69 discusses movement initially at 30°. The stated initial velocity is 120 ft/sec. Since vertical movement is associated with Sin fctn we have actual vertical velocity of $120 \sin 30° = 60$ ft/sec.

See Next Page for Clearer explanation

Now we can see the problem on page 69 is indeed very similar to the problem on page 248.

Examine Exercises on page 168. You were given the fctn s and had to find the velocity and acceleration etc. What we are now doing is in the opposite direction. Start with the end result is acceleration, V_0 and S_0 determine s.



$$\sqrt{(\Delta x)^2+(\Delta y)^2} =$$

$V_y = 120 \sin 30°$

$V_x = 120 \cos 30°$

$$\sin 30° = \frac{\Delta Y}{120}$$

$$\frac{1}{2} \cdot 120 = \Delta Y$$

$$60 = \Delta Y$$

The size of the world population in 1977 was approximately 3.8 billion. The rate of change of the population is 2%, what will be the population by the year 2000?

Start

Growth of 2% means .02 Things current level is $\frac{1}{50} \cdot Y$

$$\frac{dY}{dt} = \frac{1}{50} Y$$

$$\int \frac{dY}{Y} = \int \frac{1}{50} dt$$

$$\ln Y = \frac{1}{50} t + C$$

$$e^{\ln Y} = e^{\frac{1}{50}t + C} = e^{\frac{1}{50}t} \cdot e^{C} = Ae^{\frac{1}{50}t}$$

$1977 \rightarrow t = 0 \quad A = 3.8$

$$Y = 3.8 e^{\frac{23}{50}}$$

in 2000 $t = 23$

$$Y \approx 6 \text{ Billion.}$$

Most of the time we don't care about K. What we usually want is e^K.

Generic:

$$\frac{dY}{dt} = KY$$

$$\frac{dY}{Y} = K\,dt$$

$$\int \frac{dY}{Y} = \int K\,dt$$

$$\ln Y = Kt + C$$

$$Y = e^{Kt + C} = e^{Kt} \cdot e^{C}$$

$$Y = Ae^{Kt}$$

Here we are given the rate of growth. Thus $K > \frac{1}{50}$

a bacterial culture grows from 100 to 400 grams in 10 hours. How much was present after 3 hrs? How long will it take to double? to triple?

$Y = Ae^{kt}$

Start with 100 so $T=0, A=100$

$Y = 100e^{kt}$

when $T=10$ $Y=400$ so

$400 = 100e^{10k}$

$4 = e^{10k}$

$4^{\frac{1}{10}} = e^{k}$

$(2^2)^{\frac{1}{10}}$

$2^{\frac{1}{5}} = e^{k}$

so ....

$Y = 100(2)^{\frac{T}{5}}$

now when $T=3$

about 152 Grams of the culture.

to double

$200 = 100(2)^{\frac{T}{5}}$

$2 = 2^{\frac{T}{5}}$

$\ln 2 = \frac{T}{5}\ln 2$

$1 = \frac{T}{5}$

$T = 5$ hrs.

to triple

$300 = 100(2)^{\frac{T}{5}}$

$3 = 2^{\frac{T}{5}}$

$\ln 3 = \frac{T}{5}\ln 2$

$\frac{5\ln 3}{\ln 2} = T$

$T \approx 8$ hrs.

Consider

Carbon-14, one of the three isotopes of carbon, is radioactive and decays at a rate proportional to the amount present.

Carbon-14 has a "half-life" of 5730 yrs. That is $\frac{1}{2}$ of a given quantity will dissipate in 5730 yrs.

If there are 10 grams of Carbon-14 now, how much will be left in 2000 yrs?

$$\frac{dy}{dt} = ky$$

$$\frac{dy}{y} = k\,dt$$

$$\int \frac{1}{y}\,dy = \int k\,dt$$

$$\ln|y| = kt + C$$

$$y = e^{kt+C} = e^{kt} \cdot e^{C} = Ae^{kt}$$

$y = Ae^{kt}$ ← our usual equation.

$$\frac{1}{2}A \overset{\text{Set}}{=} Ae^{k(5730)}$$

$$\left(\frac{1}{2}\right)^{\frac{1}{5730}} = e^{k}$$

$$y = 10\left(\frac{1}{2}\right)^{\frac{t}{5730}}$$

$$y = 10\left(\frac{1}{2}\right)^{\frac{2000}{5730}}$$

$$y = 7.85 \text{ grams left}$$

In an excavations in Egypt some hair was found and tested. The scientists found that 70% of the carbon-14 expected in a living person was present in the hair. When did the person die?

$$\frac{7}{10}A = A\left(\frac{1}{2}\right)^{\frac{t}{5730}}$$

$$\left(\frac{7}{10}\right) = \left(\frac{1}{2}\right)^{\frac{t}{5730}}$$

$$\ln 7 - \ln 10 = -\frac{t}{5730}\ln 2$$

$$\frac{5730(\ln 10 - \ln 7)}{\ln 2} = t$$

$t \approx 2949$ years ago

$y = Ae^{kt}$

The set up.

$6 = 12e^{k5}$

$\frac{1}{2} = e^{5k}$ or $\left(\frac{1}{2}\right)^{\frac{1}{5}} = e^{k}$

$.871 = e^{k}$

$-\ln 2 = 5k$

$-\frac{1}{5}\ln 2 = k$

$k = -.139$

Do part b first.

$y = 12e^{-.139t}$ $\quad y = 12(.871)^{t}$

---

a) 15 days.

1.492 grams.

b) as above

c)

$10 = 12e^{-.139t}$

$\frac{5}{6} = e^{-.139t}$

$\ln 5 - \ln 6 = -.139t$

$\frac{.182}{.134} =$

1.3112 days

$10 = 12(.871)^{t}$

$\frac{5}{6} = (.871)^{t}$

$\frac{\ln 5 - \ln 6}{\ln(.871)} = t$

1.320 days.

eg 2 $1950 \to 80{,}000$

$1960 \to 100{,}000$

$1980 \to ?$ $\qquad$ $1900 \to ?$

Set up $Y = Ae^{kt}$

$Y = 80{,}000e^{kt}$

$100000 = 80{,}000e^{10k}$

$\frac{100000}{80{,}000} = e^{10k}$

$\frac{5}{4} = e^{10k}$

$\left(\frac{5}{4}\right)^{\frac{1}{10}} = e^{k}$

$1.023 = e^{k}$

$\to$ $Y = 80{,}000(1.023)^{t}$

$1980 \to t = 30$

$Y = 158{,}255.6$

$1900 \to t = -50$

$Y = 25{,}663$

Answers may vary depending on rounding & differ from 3 places, & text book.

Skip "cooling" problem

Newton's Law of Cooling is

$$\frac{dT}{dt} = K(T-a) \quad \text{①}$$

where T is Starting Temperature of an item

a is ambient temperature (constant)

K is proportionality constant related to conductivity of heat (ie insulation)

Let $u = T - a$ so u is difference in temperature

then $du = dT$

Substituting in ①

$$\frac{du}{dt} = Ku$$

Integrating $\frac{du}{u} = K\,dt$

$$\ln u = Kt + c$$

$$\vdots$$

$u = Ae^{Kt}$ when $t = 0$ $u = A$ if A is initial difference in temperature.

Given $a = 70°$ $T = 190°$ $t = 10$ New Temp $150°$

$190 - 70 = A$ u = difference $150 - 70 = 80$

$$80 = 120(e^K)^{10}$$

$$\left(\frac{2}{3}\right)^{\frac{1}{10}} = e^K = .9603$$

Now for the Problem ↓

How long will it take to cool to 120°?

it from 150° to 120°.

$150 - 70 = 80 \qquad 120 - 70 = 50$

$u = Ae^{kt} = A(.9603)^t$

$50 = 80(.9603)^t$

$\frac{5}{8} = (.9603)^t$

$\frac{\ln 5 - \ln 8}{\ln .9603} = t \ln(.9603)$

$\frac{\ln 5 - \ln 8}{\ln .9603} = t$

$11.59 = t$

so the coffee will cool to 120° in an additional 11 minutes 35 seconds.

Pg 257

9

let P = undisolved Antagonisyan

$$\frac{dP}{dt} = kP$$

$$\frac{dP}{P} = k\,dt$$

$$\ln P = kT + C$$

a) $$P = Ae^{kT}$$

b) $$.4 = 1e^{k(10)}$$

$$.4^{\frac{1}{10}} = e^{k} = .9124$$

$$.05 = (.9124)^{\tau}$$

$$\frac{\ln .05}{\ln .9124} = \tau$$

$\approx$ 33 min.

32.7 minutes.

The rate of Something: $\frac{dA}{dt}$ or $\frac{dV}{dt}$ etc

"is proportional to" means: Equals a constant times.....

① The rate of decay is proportional to the amount present

If z is amount present, then $\frac{dz}{dt}$ is rate of decay of that quantity.

So by ① $\frac{dz}{dt} = kz \Rightarrow \frac{dz}{z} = k\,dt \Rightarrow \ln z = kt + c \Rightarrow$

$z = e^{kt+c} \Rightarrow z = ce^{kt}$

The the solution for z of $\frac{dz}{dt} = kz$ is $z = ce^{kt}$.

To verify in the opposite direction

$z = ce^{kt}$

$\frac{dz}{dt} = k\,ce^{kt} = kz$

Thus. $\boxed{\frac{dz}{dt} = kz \text{ has the Solution } z = ce^{kt}.}$

Radioactive elements have a decay rate which is proportional to the amount present. This means the k abv is Negative.

[Population (of people, germs, paralyzed bets) have the same proportion with k positive.]

If $\frac{dz}{dt} = kz$ describes the rate of Radioactive decay

Then $z = ce^{kt}$ will describe the amount (z) of an item at time t. If we have z_0 amount at $t=0$, $c = z_0$ or: $z = z_0e^{kt}$. The amount of time it takes for one half of z to disappears is known as the "half life" of the substance.

This half life time period can be determined

$\frac{1}{2} z_0 = z_0 e^{kT}$ and solve for t.

$\frac{1}{2} = e^{kT}$ Notice the half life of the substance does not depend on the amount of the substance to begin with.

Eg #1 Pg 253

$V_0 = 12$

$A = 12 e^{kT}$. $\frac{1}{2}$ life is 5 days $\Rightarrow e^{5k} = \frac{1}{2}$

$6 = 12 e^{5k} \Rightarrow \frac{1}{2} = e^{5k} \Rightarrow$

$e^k = \left(\frac{1}{2}\right)^{\frac{1}{5}}$

$k = \frac{-1}{5} \ln 2 = -.1386$

a) $t = 15$

$A = 12 e^{15k} = 12 (e^k)^{15} = 12 \left(\frac{1}{2}\right)^{\frac{1}{5} \cdot 15} = 12 \left(\frac{1}{2}\right)^3 = \frac{12}{8} = \frac{3}{2}$

it at $t = 15$ and starting with 12 grams with a half life of 5 days, we will have $1\frac{1}{2}$ grams left.

b) at any time t $A = 12 e^{kT}$

$A = 12 \left(\frac{1}{2}\right)^{\frac{1}{5} \cdot T} = 12 (2)^{\frac{-T}{5}}$

c) "How long will it take for 2 grams to disappear?
if 2 are gone we have 10 left, thus.

$10 = 12 e^{kT} \Rightarrow \frac{5}{6} = \left(\frac{1}{2}\right)^{\frac{T}{5}} \Rightarrow \frac{6}{5} = 2^{\frac{T}{5}}$

Solving for T using logs you would find $T = 1.3$ days.

$\ln 6 - \ln 5 = \frac{T}{5} \ln 2$ $5(1.791 - 1.609) = 5(.1823$

$\frac{5(\ln 6 - \ln 5)}{\ln 2}$ $\frac{.9116}{\ln 2} = \frac{.9116}{.693} = 1.315$ days.

Eg 2.

T=0	1950	80,000
T=10	1960	100,000

$z = ce^{kT}$

$z = 80{,}000\, e^{kT}$

$100{,}000 = 80{,}000\, e^{10k}$

$\frac{5}{4} = e^{10k} \Rightarrow \left(\frac{5}{4}\right)^{\frac{1}{10}} = e^{k}$

$z = 80{,}000 \left(\frac{5}{4}\right)^{\frac{t}{10}}$

a) 1980 is $t = 30$

$z = 80{,}000 \left(\frac{5}{4}\right)^{\frac{1}{10} \cdot 30}$

$z = 80{,}000 \left(\frac{5}{4}\right)^{3} = 156{,}250.$

b) 1900 is $t = -50$

$z = 80{,}000 \left(\frac{5}{4}\right)^{-5} \doteq 26{,}214$

Eg #3 $\frac{dT}{dt} = k(T-A)$ let $z = T - A$ then

$\frac{dz}{dt} = \frac{dT}{dt}$ since A is constant.

Law of cooling: T = Temp at time t

A = Temp of air

$\frac{dz}{dt} = kz$ as in above problems

1st to determine k

Now $T - A = T - 70$

$t = 0$ $z = 190 - 70 = 120$ and $t = 10$ $z = 50$

$z = ce^{kt} \Rightarrow z = 120\, e^{kt}$

$80 = 120\, e^{10k}$

$\frac{2}{3} = e^{10k} \Rightarrow \left(\frac{2}{3}\right)^{\frac{1}{10}} = e^{k}.$

$z = 120 \left(\frac{2}{3}\right)^{\frac{t}{10}}$ Now $z = 120 - 70 = 50$

$50 = 120 \left(\frac{2}{3}\right)^{\frac{t}{10}}$

$\left(\frac{5}{12}\right)^{10} = \left(\frac{2}{3}\right)^{t}$ using logs as in eg FINAL ANSWER $t \approx 11\frac{1}{2}$ min.

$T = 70 - 50\left(\frac{3}{5}\right)^t$ $\quad 66.1°$

$\frac{dT'}{dt} = k(T' - A)$ $\quad$ let $u = T' - A$

$u = |T' - 70|$

$\frac{du}{dt} = ku$

$u = 50 \quad t = 0$

$u = 30 \quad t = 1$

$u = ce^{kt}$

$u = 50e^{kt}$

$30 = 50e^k$

$\frac{3}{5} = e^k$

$u = 50\left(\frac{3}{5}\right)^t$

$|T' - 70| = 50\left(\frac{3}{5}\right)^t$

$|T - 70| = 70 - T'$ $\quad \therefore \quad T' = 70 - 50\left(\frac{3}{5}\right)^t$

when $t = 5$

$T' = 70 - 50\left(\frac{3}{5}\right)^5$

$70 - 3.888$

$T = 66.112$

QUIZ ---- ANALYSIS CH 6 Name _____

NO PARTIAL CREDIT PER _____

I Differentiate (ie. Given Y, Find $\frac{DY}{DX}$) (5 pts ea)

A) $Y = e^{\frac{1}{x}}$ B) $Y = 3x^4 + 3x + 7$ C) $Y = e^3$ D) $Y = e^{x^3+1}$

E) $Y = \left(x^{\frac{1}{2}} + e^x\right)^{\frac{1}{3}}$ F) $Y = x^x$ G) $Y = e^{e^{2x}}$ H) $Y = 3e^{5^2+x^2}$

I) $Y = \frac{1}{1 + \frac{1}{x}}$ J) $Y = \frac{1}{\text{Ln } x}$ K) $Y = \text{Ln}(2x - 4)^2$ L) $Y = x^{e^x}$

II Integrate (5 pts ea)

A) $F(x) = (3x^2 + 1)^2$ B) $F(x) = e^{7x} + 1$ C) $F(x) = 10$ D) $F(x) = x^{-1}$

III Given $F(x) = (\text{Ln } x)(\text{Ln } x)(\text{Ln } x)$, Find $F'(x)$ (10 pts)

IV Find **two** distinct functions which **BOTH** have the derivative:

$F'(x) = 3x^2 + x + 2$ (10 pts)

I

a) $y = (e^{\frac{1}{x}})^2$

$y = e^{\frac{2}{x}}$

$u = \frac{2}{x}$

$\frac{du}{dx} = \frac{-2}{x^2}$

$y = e^u$

$y' = e^u \frac{du}{dx}$

a) $y' = \frac{-2e^{\frac{2}{x}}}{x^2}$

b) $y = 3x^4 + 3x + 7$

b) $y' = 12x^3 + 3$

c) $y = e^3$

c) $y' = 0$

D) $y = e^{x^3+1}$

D) $y' = 3x^2 e^{(x^3+1)}$

E) $y = (x^{\frac{1}{2}} + e^x)^{1/3}$

$y = u^{\frac{1}{3}}$

$y' = \frac{1}{3} u^{-2/3} \frac{du}{dx}$

$y' = \frac{1}{3\sqrt[3]{(x^{\frac{1}{2}} + e^x)^2}} \frac{du}{dx}$

$y' = \frac{2x^{\frac{1}{2}} e^x + 1}{6x^{\frac{1}{2}} (x^{\frac{1}{2}} + e^x)^{2/3}}$ E)

let $u = x^{\frac{1}{2}} + e^x$

$\frac{du}{dx} = \frac{1}{2x^{\frac{1}{2}}} + e^x$

f) $y = \frac{1}{1+\frac{1}{x}} = \frac{1}{\frac{x+1}{x}} = \frac{x}{x+1}$

$y' = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2}$ f)

g) $y = e^{e^{2x}}$

$y' = 2e^{2x} \cdot e^{e^{2x}}$

$y' = 2e^{2x + e^{2x}}$ g(x)

h) $y = x^x$

$\log y = x \log x$

$\frac{1}{y} \frac{dy}{dx} = \log x + 1$

h) $y' = x^x (\log x + 1)$

i) $y = \frac{1}{\log x}$

$y' = \frac{-\frac{1}{x}}{(\log x)^2} = \frac{-1}{x(\log x)^2}$ i)

J) $y = \ln(2x-4)^2$

$y = 2\ln(2x-4)$

$y' = \frac{2 \cdot 2}{2(x-2)}$

J) $y' = \frac{2}{x-2}$

K) $y = x^{e^x}$

$\log y = e^x \log x$

$\frac{1}{y} y' = \frac{e^x}{x} + e^x \log x$

K) $y' = x^{e^x} \left(\frac{e^x}{x} + e^x \log x\right)$

$y' = e^x (x^{e^x - 1} + x^{e^x} \log x)$

$y' = x^{e^x} e^x \left(\frac{1}{x} + \log x\right)$

$y' = x^{e^x - 1} e^x (1 + x \log x)$

l) $y = 8e^{5^2 + x^2}$ $6e^{5^2+x^2}$

l) $y' = 2ax e^{b^2 + x^2}$

II $\int f(x) = \int (x+1)^2$

a) $= \int x^2 + 2x + 1$

a) $= \frac{x^3}{3} + x^2 + x + C$

b) $\int f(x) = \int e^{7x} + 1$

B $= \frac{1}{7} e^{7x} + x + C$

c) $\int f(x) = \int 10$

E) $= 10x + C$

D) $\int f(x) = \int x^{-1}$

D) $= \ln|x| + C$

$$f(x) = \frac{1}{2}\ln\left(\frac{x-1}{x+1}\right)$$

$$= \frac{1}{2}\ln(x-1) - \frac{1}{2}\ln(x+1)$$

$$f' = \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$$

$$\frac{x+1-x+1}{2(x^2-1)} = \frac{1}{x^2-1}$$

$$f(x) = \frac{1}{2}\ln\left(\frac{x-1}{x+1}\right)$$

$$x = \frac{1}{2}\ln\frac{(z-1)}{(z+1)}$$

$$e^{2x} = \frac{z-1}{z+1}$$

$$e^{2x}z + e^{2x} = z - 1$$

$$(e^{2x}-1)z = -(1+e^{2x})$$

$$z = -\frac{1+e^{2x}}{e^{2x}-1}$$

$$z = \frac{1+e^{2x}}{1-e^{2x}}$$

III $u(t) = (\ln t)(\ln t)(\ln t)$

$u(t) = (\ln t)^3$

let $J = \ln t$ $\quad \frac{dJ}{dt} = \frac{1}{t}$

$u(J) = J^3$

$\frac{du}{dJ} = 3J^2 \Rightarrow 3(\ln t)^2$

$\frac{du}{dt} = \frac{3(\ln t)^2}{t}$ **III**

IV $f(x) = \ln x$

$\frac{df(x)}{dx} = \frac{1}{x}$

$df(x) = \frac{1}{x}\,dx \Big]_{x=2}$

$= \frac{1}{2} \cdot \left(\frac{-2}{10}\right)$ $dx = -0.2$

IV $= -\frac{1}{10}$

V $d(a) = 3x^2 + x + 2$

$a \Rightarrow x^3 + \frac{1}{2}x^2 + 2x + 1$ **V**

$d(b) = 3x^2 + x + 2$

$b \Rightarrow x^3 + \frac{1}{2}x^2 + 2x + 2$

VI $f(x) = \frac{1}{2}\log\left(\frac{x-1}{x+1}\right)$

$x = \frac{1}{2}\log\left(\frac{z-1}{z+1}\right)$

$e^{2x} = \frac{z-1}{z+1}$

$e^{2x}(z+1) = z-1$

$e^{2x}z + e^{2x} = z - 1$

$e^{2x}z - z = -(e^{2x}+1)$

$z = \frac{-(e^{2x}+1)}{e^{2x}-1} = f^{-1}(x)$ **VI**

5··15··80

graded very separately. Splits based on partial credit. Not even for minor errors except for last problem. Balanced over minor errors thus.

85· +
75· +
65· +
60· +
60· +
55· +
50· +
50· +
45· +
35· +
35· +
30· +
25· +
25· +
25· +
25· +
20· +
15· +
15· +
15· +
810· T

810· +

20 ◇C

20· =
40·5

(9) $N = a_0 e^{kT}$ $\quad 3 \cdot 10^5 = 10^5 e^{2k}$

$e^{2k} = 3$

$e^k = \sqrt{3}$ $\therefore$

$N = 10^5 (\sqrt{3})^T$

a) $T = 1$ $\therefore$ $N = 10^5 \sqrt{3}$

b) $T = 3$ $\therefore$ $N = 10^5 (3)^{3/2}$.

I

A) $y = e^{\frac{1}{x}}$

$y' = -\frac{1}{x^2} e^{\frac{1}{x}}$

B) $y = 3x^4 + 3x + 7$

$y' = 12x^3 + 3$

C) $y = e^3$

$y' = 0$

D) $y = e^{x^3+1}$

$y' = 3x^2 e^{x^3+1}$

E) $y = \left(x^{\frac{1}{2}} + e^x\right)^{\frac{1}{3}}$

$y' = \frac{1}{3}\left(x^{\frac{1}{2}} + e^x\right)^{-\frac{2}{3}}\left[\frac{1}{2}x^{-\frac{1}{2}} + e^x\right]$

F) $y = x^x$

$y = e^{x \ln x}$

$y' = x^x(\ln x + 1)$

G) $y = e^{e^{2x}}$

$y' = e^{e^{2x}}\left[e^{2x} \cdot 2\right]$

$= 2e^{2x} e^{e^{2x}}$

H) $y = 3e^{5^2 + x^2}$

$y' = 6x e^{5^2 + x^2}$

I) $y = \dfrac{1}{1 + \frac{1}{x}}$

$= \dfrac{1}{\frac{x+1}{x}}$

$y = \dfrac{x}{x+1}$

$y' = \dfrac{x+1-x}{(x+1)^2}$

$= \dfrac{1}{(x+1)^2}$

J) $y = \dfrac{1}{\ln x}$

$= (\ln x)^{-1}$

$y' = -(\ln x)^{-2}\frac{1}{x}$

$= \dfrac{-1}{x(\ln x)^2}$

K) $y = \ln(2x-4)^2$

$= 2\ln(2x-4)$

$y' = \dfrac{2 \cdot 2}{2(x-2)}$

$= \dfrac{2}{x-2}$

L) $y = x^{e^x}$

$y = e^{e^x \ln x}$

$y' = x^{e^x}\left[e^x \ln x + \dfrac{e^x}{x}\right]$

II

A) $\int (3x^2+1)^2\,dx$

$\int 9x^4+6x^2+1\,dx$

$\frac{9x^5}{5}+2x^3+x+c$

B) $\int e^{7x}+1\,dx$

$\frac{1}{7}e^{7x}+x+c$

c) $f(x)=\int 10\,dx$

$10x+c$

D) $\int x^{-1}\,dx = \ln|x|+c$

III $f(x) = (\ln x)^3$

$= \frac{3(\ln x)^2}{x}$

IV $f'(x) = 3x^2+x+2$

$f(x) = x^3+\frac{x^2}{2}+2x+1$

OR $f(x) = x^3+\frac{x^2}{2}+2x+2$

$\cos\left(\frac{\pi}{4}+x\right)$

① $\cos\left(\frac{\pi}{4}+\frac{\pi}{3}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{3} - \sin\frac{\pi}{4}\sin\frac{\pi}{3}$

$\frac{\sqrt{2}}{2}\frac{1}{2} - \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} = \frac{\sqrt{2}-\sqrt{6}}{4}$

② $\cos\left(\frac{\pi}{4}+\frac{\pi}{2}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{2} - \sin\frac{\pi}{4}\sin\frac{\pi}{2}$

$0 \quad - \frac{\sqrt{2}}{2}\cdot 1 = -\frac{\sqrt{2}}{2}$

③ $\cos\left(\frac{\pi}{4}+\frac{2\pi}{3}\right) = \cos\frac{\pi}{4}\cos\frac{2\pi}{3} - \sin\frac{\pi}{4}\sin\frac{2\pi}{3}$

$\frac{\sqrt{2}}{2}\left(-\frac{1}{2}\right) - \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} = -\frac{\sqrt{2}-\sqrt{6}}{4}$

④ $\cos\left(\frac{\pi}{4}+\frac{5\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{5\pi}{6} - \sin\frac{\pi}{4}\sin\frac{5\pi}{6}$

$\frac{\sqrt{2}}{2}\left(-\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{2}-\sqrt{6}}{4}$

$\cos\left(\frac{\pi}{3}-x\right)$

⑤ $\cos\left(\frac{\pi}{3}-\frac{\pi}{6}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{6} + \sin\frac{\pi}{3}\sin\frac{\pi}{6}$

$\frac{1}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\frac{1}{2} = \frac{\sqrt{3}}{2}$

⑥ $\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}$

$\frac{1}{2}\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} = \frac{\sqrt{2}+\sqrt{6}}{4}$

⑦ $\cos\left(\frac{\pi}{3}-\frac{\pi}{2}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{2} + \sin\frac{\pi}{3}\sin\frac{\pi}{2}$

$\frac{1}{2}\quad 0 \quad + \frac{\sqrt{3}}{2}\cdot 1 = \frac{\sqrt{3}}{2}$

⑧ $\cos\left(\frac{\pi}{3}-\frac{2\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

⑨ $\cos\left(\frac{\pi}{3}-\frac{5\pi}{6}\right) = \cos\frac{\pi}{3}\cos\frac{5\pi}{6} + \sin\frac{\pi}{3}\sin\frac{5\pi}{6}$

$\frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2}\frac{1}{2} = 0$

(10) $\cos\left(\frac{\pi}{3} - \frac{3\pi}{4}\right) = \cos\frac{\pi}{3}\cos\frac{3\pi}{4} + \sin\frac{\pi}{3}\sin\frac{3\pi}{4}$

$\frac{1}{2} \cdot \frac{-\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$

$\cos x_2 = \frac{3}{5}$ $\quad \cos x_1 = \frac{5}{13}$

$\sin x_2 = \pm\frac{4}{5}$ $\quad \sin x_1 = \frac{\pm 12}{13}$

(11) $\cos(x_1 - x_2) = \cos x_1 \cos x_2 + \sin x_1 \sin x_2$

$\frac{3}{5}\ \frac{5}{13}\quad \frac{4}{5}\ \frac{12}{13}$

$\frac{15 + 48}{65} = \frac{63}{65}$

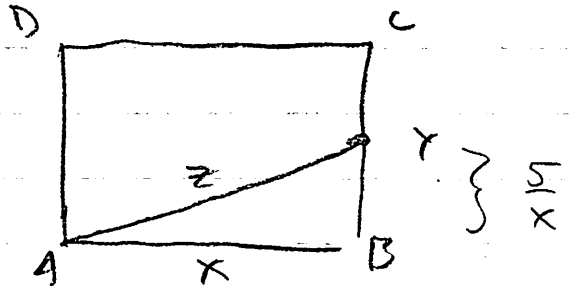
(12) $\cos(x_2 - x_1) = \text{same as 11}$ $\quad \frac{63}{65}$

(13) $\cos\left(\frac{\pi}{4} + x_2\right) \Rightarrow \cos\frac{\pi}{4}\cos x_2 - \sin\frac{\pi}{4}\sin x_2$

$\frac{\sqrt{2}}{2}\ \frac{3}{5} - \frac{\sqrt{2}}{2}\ \frac{4}{5} = \frac{-\sqrt{2}}{10}$

(14)

XV



$xy = 10$

$y = \frac{10}{x}$

$$z = \sqrt{x^2 + \frac{25}{x^2}}$$

$$= \sqrt{\frac{x^4+25}{x^2}}$$

$$z = \frac{\sqrt{x^4+25}}{x} \quad u \quad v$$

$\frac{dz}{dx}$

$$z' = \frac{x \cdot \frac{4x^3}{\sqrt{x^4+25}} - \sqrt{x^4+25}}{x^2}$$

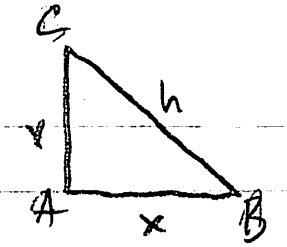
$$\frac{4x^4 - (x^4+25)}{x^2\sqrt{x^4+25}}$$

$$\frac{3x^4-25}{x^2\sqrt{x^4+25}} \overset{\text{Set}}{=} 0$$

$$x^4 = \frac{25}{3}$$

$$x = \sqrt[4]{\frac{25}{3}}$$

X



$\frac{dx}{dt} = 2$ $\quad$ $x^2 + y^2 = h^2$

$\frac{dy}{dt} = -3$ $\quad$ $x\frac{dx}{dt} + y\frac{dy}{dt} = h\frac{dh}{dt}$

$\frac{dh}{dt} = ?$ $\quad$ $6 \cdot 2 + 8(-3) = 10\frac{dh}{dt}$

$\frac{12 - 24}{10} = \frac{dh}{dt}$

$-\frac{12}{10} = -\frac{6}{5} = \frac{dh}{dt}$

Bonus

$A = \frac{1}{2} x \cdot y$

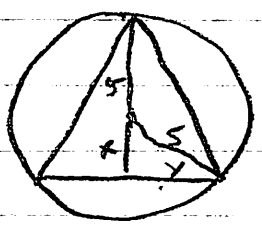
$\frac{dA}{dt} = ?$ $\quad$ $\frac{dA}{dt} = \frac{1}{2} y \frac{dy}{dt} + \frac{1}{2} x \frac{dx}{dt}$

$= \frac{1}{2} 8(-3) + \frac{1}{2}(6)2$

$= -12 + 6$

$\frac{dA}{dt} = -6$ Area getting smaller.

XI



$A = \frac{1}{2} y(x+5)$ $\quad$ $y = \sqrt{5-x^2}$

$= \frac{1}{2} \underset{u}{(x+5)} \underset{v}{\sqrt{5-x^2}}$

$A' = \frac{1}{2}\sqrt{5-x^2} + \frac{1}{2}(x+5)\frac{-x}{\sqrt{5-x^2}} = 0$

$5 - x^2 - x^2 - 5x = 0$

$2x^2 + 5x - 5 = 0$

$\therefore$

I

A) $y = e^{\frac{1}{x}}$

$y' = -\frac{1}{x^2} e^{\frac{1}{x}}$

B) $y = 3x^4 + 3x + 7$

$y' = 12x^3 + 3$

C) $y = e^3$

$y' = 0$

D) $y = e^{x^3+1}$

$y' = 3x^2 e^{x^3+1}$

E) $y = \left(x^{\frac{1}{2}} + e^x\right)^{\frac{1}{3}}$

$\frac{dy}{dx} = \frac{1}{3}\left(x^{\frac{1}{2}} + e^x\right)^{-\frac{2}{3}} \left[\frac{1}{2x^{\frac{1}{2}}} + e^x\right]$

F) $y = x^x$

$\ln y = x \ln x$

$\frac{1}{y}\frac{dy}{dx} = \ln x + 1$

$\frac{dy}{dx} = x^x(1 + \ln x)$

G) $y = e^{e^{2x}}$

$y' = 2e^{2x} e^{e^{2x}}$

H) $y = 3e^{5^2+x^2}$

$y' = 6x e^{5^2+x^2}$

I) $y = \frac{1}{1+\frac{1}{x}}$

$= \frac{1}{\frac{x+1}{x}}$

$= \frac{x}{x+1}$ u v

$y' = \frac{x+1-x}{(x+1)^2}$

$= \frac{1}{(x+1)^2}$

J) $y = \frac{1}{\ln x}$

$= (\ln x)^{-1}$

$y' = -(\ln x)^{-2}\frac{1}{x}$

$= -\frac{1}{x(\ln x)^2}$

K) $y = \ln(2x-4)^2$

$y = 2\ln(2x-4)$

$y' = \frac{2 \cdot 2}{2(x-2)}$

$= \frac{2}{x-2}$

L) $y = x^{e^x}$ u v

$\ln y = e^x \ln x$

$\frac{1}{y}\frac{dy}{dx} = e^x \ln x - \frac{e^x}{x}$

$\frac{dy}{dx} = x^{e^x}\left[x^{e^x} - \frac{e^x}{x}\right]$

II

A) $\int (3x^2+1)^2\,dx$

$\int 9x^4+6x^2+1\,dx$

$\frac{9x^5}{5}+2x^3+x+C$

B) $\int e^{7x}+1\,dx$

$\frac{1}{7}e^{7x}+x+C$

C) $\int 10\,dx$

$10x+C$

D) $\int \frac{1}{x}\,dx = \ln|x|+C$

III $f(x) = (\ln x)^3$

$f'(x) = \frac{3(\ln x)^2}{x}$

IV $\int 3x^2+x+2\,dx$

$f(x) = x^3+\frac{1}{2}x^2+2x+1$

$f(x) = x^3+\frac{1}{2}x^2+2x-1$